On Analysis and Control of Interconnected Finite- and Infinite-dimensional Physical Systems.

Dutch Institute of Systems and Control

Geometric Network Modeling and Control of Complex Physical Systems

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ON ANALYSIS AND CONTROL OF INTERCONNECTED FINITE- AND INFINITE-DIMENSIONAL PHYSICAL SYSTEMS

DISSERTATION

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prof. dr. W.H.M. Zijm,
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Summary

This thesis is aimed at the analysis, control and simulation of complex physical systems from different domains. We use the recently developed framework of port-Hamiltonian systems which formalizes the interconnection structure of the system through a geometric object called a Dirac structure. It represents the system dynamics as a generalized Hamiltonian system and provides powerful tools for analysis and control. In this book we discuss three main issues, namely interconnections of systems from different physical domains, control of systems by interconnection, and spatial discretization of infinite-dimensional port-Hamiltonian systems.

We have studied interconnections of port-Hamiltonian systems by studying the composition of their Dirac structures. Since the interconnection is power conserving, the interconnected system is energy conserving and can again be described as a port-Hamiltonian system. We study interconnections in the case of finite-dimensional systems, as well as interconnections of infinite-dimensional Dirac structures as appearing in the port-Hamiltonian formulation of conservation laws. We also extend these results to study interconnections of port-Hamiltonian systems with dissipation. Finally we study the interesting case where we interconnect finite and infinite-dimensional systems. The total interconnection defines again a port-Hamiltonian system which we call a mixed finite- and infinite-dimensional port-Hamiltonian system.

We have derived explicit formulas for the set of achievable Dirac structures (by composition of a given plant Dirac structure with a to-be-designed controller Dirac structure). This has led to a characterization of the achievable Casimir functions, and applications towards stabilization have been developed. Next this theory of achievable Dirac structures and hence the characterization of achievable Casimirs has been successfully generalized to infinite-dimensional systems, and also to the mixed finite-dimensional and infinite-dimensional case.

An important question, from the control and simulation point of view, is how to define a discretization procedure for an infinite-dimensional port-Hamiltonian system that retains the physical structure of the system? We answer this question by presenting a discretization procedure in the port-Hamiltonian framework, by defining special approximating objects which serve as the discrete analogue of various differential forms. We show that the result-

ing finite-dimensional system has port-Hamiltonian structure and it retains all the physical properties of its infinite-dimensional counterpart.

Samenvatting

Dit proefschrift behandelt analyse, regeling, en simulatie van complexe systemen uit verscheidende fysische domeinen. We gebruiken het nieuwe formalisme van poort-Hamiltonianen, dat de interconnectie-structuur van een systeem weergeeft als een meetkundig object, bekend als de Diracstructuur. Hierdoor verwordt de systeemdynamica tot een veralgemeniseerde Hamiltoniaanse dynamica met krachtige hulpmiddelen voor analyse en regeltechniek. In deze thesis behandelen we drie hoofdzaken: de interconnecties van systemen met verschillende fysische achtergronden, de regeltheorie van systemen met behulp van interconnecties, en de ruimtelijke discretisatie van oneindig dimensionale poort-Hamiltoniaanse systemen.

Wij bestudeerden de interconnecties van poort-Hamiltoniaanse systemen door de samenstelling van hun Diracstructuur. Vanwege het vermogensbehoud van de interconnectie, is de energie behouden in het samengestelde, of interconnecte, systeem, en het geheel kan weer beschreven worden als een poort-Hamiltoniaans systeem. Beide, de eindig dimensionale en de oneindig dimensionale systemen, zijn onderzocht, waarbij de Diracstructuur van de laatste het resultaat is van de poort-Hamiltoniaanse beschrijving van behoudswetten. Verder is de beschrijving uitgebreid tot systemen met energieverlies. Als laatste is het interessante geval bestudeerd waar een eindig en een oneindig systeem aan elkaar verbonden is met een interconnectie. Het geheel noemen wij een gemengd eindig en oneindig poort-Hamiltoniaans systeem.

Wij hebben vergelijkingen afgeleid voor de verzameling van de realiseerbare Diracstructuren (door samenstelling van een gegeven Diracstructuur met dat van een, te ontwerpen, regelende Diracstructuur). Dat leidde tot een beschrijving van realiseerbare Casimirfuncties, met als resultaat methoden voor stabilisatie. Deze theorie van realiseerbare Diracstructuren en hun realiseerbare Casimirfuncties is uitgebreid naar oneindig dimensionale, en gemengd eindig-oneindig dimensionale gevallen.

Een belangrijke vraag, met betrekking tot het regelen en simuleren, is hoe een oneindig dimensionaal poort-Hamiltoniaans systeem kan worden gediscretiseerd zodat de fysische eigenschappen behouden blijven? Ons antwoord is een poort-Hamiltoniaanse discretisatie methode, waarbij speciale objecten bij benadering de verschillende differentialen kunnen vervangen. Wij hebben laten zien dat het resulterende eindig dimensionale systeem de poort-Hamiltoniaanse structuur en alle fysische eigenschappen van de oneindig di-

mensionale tegenhanger behoudt.

Notation

Symbol	Description	Page
$\mathcal{L}$	the Lagrangian	1
$\mathcal{F}$	the space of flow variables	13
$\mathcal{F}^*$	the space of effort variables	13
f	a flow vector	13
e	an effort vector	13
H	the Hamiltonian for a finite-dimensional system	2
$\mathcal{H}$	the Hamiltonian for an infinite-dimensional system	29
$\mathcal{D}$	a Dirac structure	14
$\mathcal{DR}$	a Dirac structure with dissipation	77
$\mathcal{D}_\infty$	an infinite-dimensional Dirac structure	57

$\mathcal{C}$	a Casimir function	72
$\|$	composition operator for Dirac structures and resistive relations	45
Z	represents the spatial domain of an infinite-dimensional system	28
∂Z	represents the boundary of an infinite-dimensional system	28
$\Omega^k(Z), \Omega^k(\partial Z)$	space of exterior forms on an n-dimensional manifold Z, respectively ∂Z	28
d	exterior derivative mapping $k-$forms on Z to $k+1-$forms on Z	29
$*$	the Hodge star operator mapping $k-$forms on Z into $n-k-$forms on Z	32
$\wedge$	wedge product of differential forms	32

1

Introduction

"Get the Physics right, rest is Mathematics." - Rudolph Kalman.

In this thesis we present some results contributing towards analysis and control of finite and infinite-dimensional port-Hamiltonian systems. We begin with a few results from the literature which would serve as a background for the rest of the thesis. We also present a couple of examples to make the reader familiar with the class of systems we deal in this piece of work. A few concepts or ideas that might appear vague or unexplained will be explained in the following chapters. Towards the end of this chapter we summarize the results obtained in the rest of the thesis.

To begin we show how the well known Hamiltonian equations can be generalized to a class of systems called port-Hamiltonian systems. We also formalize the notion of power-conserving interconnections which leads to the definition of an implicit port-Hamiltonian system.

1.1 From Euler-Lagrange to port-Hamiltonian systems

For a system with n degrees of freedom the Euler-Lagrange equations are defined by a Lagrangian function $\mathcal{L}(q, \dot{q})$, where $q = (q_1, ..., q_n) \in \mathbb{R}^n$ are the configuration variables and are given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}(q(t), \dot{q}(t))\right) - \frac{\partial \mathcal{L}}{\partial q}(q(t), \dot{q}(t)) = \tau, \tag{1.1}$$

where d/dt represents the total time derivative and $\tau = (\tau_1, ..., \tau_n)^T$ is the vector of generalized forces acting on the system. These equations can be derived as a first order condition of a variational principle. The Lagrangian

$\mathcal{L}(q, \dot{q})$ equals $K(q, \dot{q}) - V(q)$, where $K(q, \dot{q})$ represents the kinetic energy (co-energy) of the system and $V(q)$ the potential energy of the system. $\frac{\partial \mathcal{L}}{\partial \dot{q}}$ denotes the column vector of partial derivatives of $\mathcal{L}(q, \dot{q})$ with respect to the generalized velocities $\dot{q}_1, ..., \dot{q}_n$ and similarly for $\frac{\partial \mathcal{L}}{\partial q}$. In standard mechanical systems the kinetic energy K is of the form

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q},$$

where the $n \times n$ inertia (generalized mass) matrix $M(q)$ is symmetric and positive definite for all q. In this case the vector of generalized momenta $p = (p_1, ..., p_n)^T$, defined for any Lagrangian $\mathcal{L}$ as $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, is simply given by

$$p = M(q)\dot{q}.$$

Equation (1.1) is called the Euler-Lagrange equation which describes the equations of motion of a system. Later what William Rowan Hamilton (1805-1865) did was to rewrite the second order equation (1.1) into a set of first order equations using the function $H(q, p)$ obtained by the Legendre transform of $\mathcal{L}(q, \dot{q})$, defined by

$$H(q, p) = p^T \dot{q} - \mathcal{L}(q, \dot{q}),$$

where $\dot{q}$ is expressed as a function of q and p through the equation

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}.$$

It is then easy to see that the Euler Lagrange equation (1.1) can be written as a set of first order Hamiltonian equations

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q(t), p(t)) \quad (= M^{-1}(q(t))p(t)) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q(t), p(t)) + \tau, \\ y &= \tfrac{\partial H}{\partial p}(q(t), p(t)). \end{aligned} \tag{1.2}$$

y denotes the output of the system and is defined such that the product of the input τ and the output y has dimensions of power. Observe that in the above equations, apart from them being first-order equations, the Hamiltonian H in most cases describes the total energy of the system. For a mechanical system, the Hamiltonian is the sum of the kinetic and the potential energies, $H(q, p) = \frac{1}{2}p^T M^{-1} p + V(q)$. The following energy balance immediately follows from (1.2)

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial^T H}{\partial q}(q, p)\dot{q} + \frac{\partial^T H}{\partial p}(q, p)\dot{p} \\ &= \frac{\partial^T H}{\partial p}(q, p)\tau = \dot{q}^T \tau, \end{aligned}$$

expressing that the increase in energy of the system is equal to the supplied work, which shows conservation of energy. System (1.2) is an example of a Hamiltonian system with collocated inputs and outputs, which more generally is given in the following form

$$\begin{aligned}
\dot{q} &= \tfrac{\partial H}{\partial p}(q,p), \quad (q,p) = (q_1, ..., q_n, p_1, ..., p_n) \\
\dot{p} &= \tfrac{\partial H}{\partial q}(q,p) + B(q)u, \quad u \in \mathbb{R}^m, \\
y &= B^T(q)\tfrac{\partial H}{\partial p}(q,p) \quad (= B^T(q)\dot{q}), \quad y \in \mathbb{R}^m,
\end{aligned} \tag{1.3}$$

where $B(q)$ is the input force matrix, with $B(q)u$ denoting the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. The state space of (1.3) with local coordinates (q,p) is usually called the phase space. In case $m < n$ we speak of an underactuated system. If $m = k$ and the matrix $B(q)$ is everywhere invertible, then the Hamiltonian system is called fully actuated. Because of the form of the output equations $y = B^T(q)\dot{q}$ we again obtain the energy balance

$$\frac{dH}{dt}(q,p) = u^T y.$$

Hence if H is non-negative (or, bounded from below), any Hamiltonian system (1.3) is a lossless state space system.

Later in 1992 Bernhard Maschke and Arjan van der Schaft [30] generalized the class of Hamiltonian system (1.3) into systems (called *port-Hamiltonian systems*) which are described in local coordinates as

$$\begin{aligned}
\dot{x} &= J(x)\tfrac{\partial H}{\partial x}(x) + g(x)u, \quad x \in \mathcal{X}, u \in \mathbb{R}^m \\
y &= g^T(x)\tfrac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m.
\end{aligned} \tag{1.4}$$

Here $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be skew-symmetric

$$J(x) = -J^T(x), \tag{1.5}$$

and $x \in (x_1, ..., x_n)$ are local coordinates for an $n-$dimensional state space manifold $\mathcal{X}$. Because of (1.5) we get the energy balance

$$\frac{dH}{dt}(x(t)) = u^T(t)y(t).$$

This shows that (1.4) is lossless if $H \geq 0$. The system (1.4) together with (1.5) is called a port-Hamiltonian system with structure matrix $J(x)$ and Hamiltonian H. Note that (1.3) (and hence (1.2)) is a particular case of (1.4) with $x = (q,p)$, and $J(x)$ being given by a constant skew-symmetric matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $g(q,p) = \begin{bmatrix} 0 \\ B(q) \end{bmatrix}$.

1.2 From network modeling to port-Hamiltonian systems

In the port-based network models of complex physical systems, the overall system is seen as interconnection of energy-storing elements via basic interconnection (balance) laws as Newton's third law or Kirchhoff's laws, as well as power-conserving elements like transformers, kinematic pairs and ideal constraints, together with energy-dissipating elements. The idea is then to formalize the basic interconnection laws together with the power-conserving elements by a geometric structure and to define the Hamiltonian as the total energy stored in the system.

In order to define power, we start with a finite-dimensional linear space and its dual. Let $\mathcal{F}$ be an $l-$dimensional linear space and denote its dual (the space of linear functions on $\mathcal{F}$) by $\mathcal{F}^*$. The product space $\mathcal{F} \times \mathcal{F}^*$ is considered to be the space of power variables, with power defined by

$$P =< f^* \mid f >, \quad (f, f^*) \in \mathcal{F} \times \mathcal{F}^*,$$

where $< f \mid f^* >$ denotes the duality product, that is the linear function $f^* \in \mathcal{F}$ acting on $f \in \mathcal{F}$. Often we call $\mathcal{F}$ the space of flows f and $\mathcal{F}^*$ the space of efforts e, with power of an element $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $< e \mid f >$. A Dirac structure can then be defined as a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that $\mathcal{D} = \mathcal{D}^\perp$ with respect to the symmetric bilinear form defined by

$$< (f_1, e_1), (f_2, e_2) >_{\mathcal{F} \times \mathcal{F}^*} := < e_2 \mid f_1 > + < e_1 \mid f_2 > .$$

Since, for all $(f, e) \in \mathcal{D}$, $< e \mid f >= 0$, a Dirac structure $\mathcal{D}$ defines a power-conserving relation between the power variables $(e, f) \in \mathcal{F} \times \mathcal{F}^*$. We will elaborate more on this in Chapter 2.

1.2.1 Physical models

A physical system is described by a set of energy-storing elements, a set of energy-dissipating or resistive elements, and a set of ports, by which interaction with the environment can take place. These elements are interconnected to each other by a power-conserving interconnection, see Figure 1.1. We now present a short description of these elements.

Energy storage elements: A storage element is an element with the property of storing energy. Typical examples are masses, springs, capacitors or inductors. Every energy storage element is characterized by an input signal $u(t)$, an output signal $y(t)$, a state variable $x(t)$ and an energy function $H(x)$. The mathematical model is given by

$$\begin{aligned} \dot{x} &= u(t) \\ y(t) &= \frac{\partial H}{\partial x}(x). \end{aligned} \tag{1.6}$$

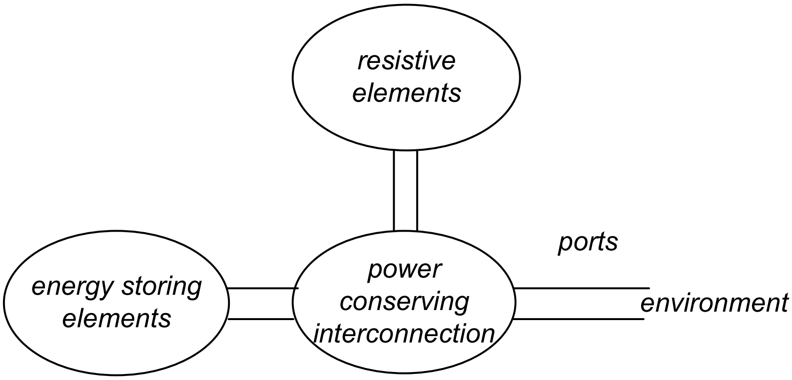


Figure 1.1: Implicit port-Hamiltonian system

The above definition of u and y corresponds to a conjugate pair of power variables as seen in the expression below:

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} = yu = P_s,$$

meaning that the variation of internal energy equals the power P_s supplied through the port.

Energy dissipating elements: An energy dissipating element models the irreversible phenomena of the conversion of (mechanical, electrical, etc.) energy to thermal one. An energy dissipating element is characterized by a statical relation between effort and flow variables

$$e = Z(f) \text{ (impedance form) or } f = Y(e) \text{ (admittance form),}$$

for which the following inequalities have to hold

$$Z(f)f \leq 0 \text{ or } eY(e) \leq 0.$$

This implies that

$$P = ef \leq 0.$$

Similarly power-conserving elements like transformers and gyrators can be added to the conservation laws in order to define the Dirac structure.

Implicit port-Hamiltonian systems: An implicit Hamiltonian system is then defined as [58],

$$(-\dot{x}(t), \frac{\partial H}{\partial x}(x), f_R, e_R, f, e) \in \mathcal{D}, \tag{1.7}$$

where $(-\dot{x}(t), \frac{\partial H}{\partial x}(x))$ are the flows and efforts corresponding to the energy-storing elements, (f_R, e_R) the flows and efforts corresponding to the energy

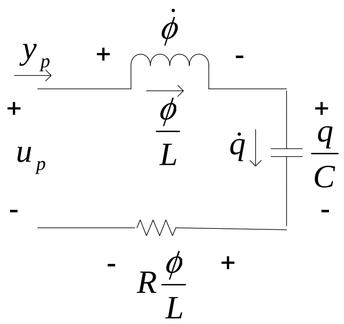


Figure 1.2: The series RLC circuit

dissipating elements, (f, e) the ports available for interaction with the environment. $\mathcal{D}$ is the underlying Dirac structure and H is called the Hamiltonian function.

Example 1.1. Consider the series RLC circuit as shown in Figure 1.2. The dynamics are given by the Kirchhoff's voltage and current laws as follows

$$u_p = \dot{\phi} + \tfrac{q}{C} + R\tfrac{\phi}{L}$$

$$y_p = \tfrac{\phi}{L} = \dot{q}.$$

This can be written as a port-Hamiltonian system as follows: The state variables are $x = [q, \phi]^T$, the charge in the capacitor and the flux in the inductor. R,L and C respectively correspond to the values of the resistor, inductor and capacitor. The total energy of the system is given by

$$H = \frac{1}{2}\left(\frac{q^2}{C} + \frac{\phi^2}{L}\right).$$

$(-\dot{q},-\dot{\phi})$, which represent the current through the capacitor and voltage across the inductor respectively, are the flow variables corresponding to the energy-storing elements. Similarly, $(\frac{q}{C}, \frac{\phi}{L})$ which represent the voltage across the capacitor and the current through the inductor, are the effort variables corresponding to the energy-dissipating elements. We can then write the series RLC circuit as a system of the form (1.4) with dissipation as

$$\begin{bmatrix} \dot{q} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} = \frac{q}{C} \\ \frac{\partial H}{\partial \phi} = \frac{\phi}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_p$$

$$y_p = \frac{\partial H}{\partial \phi} = \frac{\phi}{L}.$$

The terms (f, e) in (1.7) correspond to the port variables, which in this case are the voltage u_p and the current y_p respectively. Similarly (f_R, e_R) in (1.7) respectively correspond to the voltages and current in the resistance. The Dirac structure in this case corresponds to Kirchhoff's voltage and current laws.

1.3 Interconnection and Control

A power-conserving interconnection describes an interconnection between systems in such a way that there is no gain or loss of energy in the interconnection. Since port-Hamiltonian systems include the definition of external variables as being power variables, it is natural to consider interconnections of such systems with other port-Hamiltonian systems. Since the interconnections are power-conserving, the interconnected system is again an energy-conserving system. It has been shown in [7, 59, 12, 42] that the power-conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system. The total Dirac structure is then the composition of the individual Dirac structures and the Hamiltonian of the interconnected system is the sum of the Hamiltonians of the subsystems. The property of interconnection of systems is useful when modeling energy-conserving systems using a modular approach, where the system is thought of as the interconnection of a number of subsystems. In Chapter 3 we present a few results towards interconnections of systems both from the finite and infinite-dimensional domain.

The property of interconnection is also important from the control point of view of port-Hamiltonian systems, as control can be seen as in a natural way as interconnection of a (given plant) system with other subsystems (controller systems). Since the plant and the controller systems both have the port-Hamiltonian structure, the resulting closed-loop system is also a port-Hamiltonian system. This allows the application of energy-shaping (or passivity based) methods for control of port-Hamiltonian systems. The idea is to shape the energy of the system by adding a controller system and also possibly add damping to ensure asymptotic stability. It has been shown in the literature [38] that passivity based techniques have been very instrumental in the control of physical systems, defined as Euler-Lagrange or Hamiltonian systems and it could thus be expected that this approach is equally powerful for control of port-Hamiltonian systems.

In this thesis we are interested in the stabilization by Casimir generation, also called the Energy-Casimir method, of port-Hamiltonian systems. The key in this method is to use in the Lyapunov analysis next to the Hamiltonian additional conserved quantities (called *Casimirs*) which may be present in the system. Casimirs are functions that are conserved quantities of the system for every Hamiltonian. If we can find Casimirs for a system then the candidate Lyapunov functions can be sought within the class of combination of the Hamiltonian H and the corresponding Casimir function. Consider now the problem of designing a controller port-Hamiltonian system such that the closed-loop system has the desired stability properties. Suppose we want to stabilize the plant port-Hamiltonian system around a desired equilibrium x_*. In case x_* is not a minimum for H_P, the plant Hamiltonian, then a possible strategy is to generate Casimir functions for the closed-loop system by choosing an appropriate controller port-Hamiltonian system. We then generate candidate Lyapunov functions for the closed-loop system as the sum of the plant and controller Hamiltonian systems and the corresponding Casimir functions. This strategy is based on finding all the achievable Casimirs of the closed-loop system. Furthermore, since the closed-loop Casimirs are based on the closed-loop Dirac structures, the problem reduces to finding all the achievable closed-loop Dirac structures. Some results in this direction have been obtained in [7, 42] and in Chapter 4 and 5, we discuss the concepts on Casimirs and control in more detail.

1.4 Infinite-dimensional port-Hamiltonian systems

The framework of port-Hamiltonian systems has also been extended to infinite-dimensional systems (see [61, 60] for example) such as Maxwell's equations incorporating energy radiation though the boundary, the n-dimensional wave equation, fluid dynamical systems and so on. Hereto a special type of infinite-dimensional Dirac structure has been introduced, based on the Stokes' theorem which is called the Stokes-Dirac structure. Physically, the Stokes-Dirac structure captures the basic balance laws of the system, like Faradays and Ampere's laws or mass balance. The port-Hamiltonian formulation of infinite-dimensional systems is a non-trivial extension of the Hamiltonian formulation of partial differential equations by means of [34], since in the later case it is crucially assumed that the boundary conditions are such that the energy flow through the boundary of the spatial domain is zero.

Example 1.2. Consider the flow of water through an open-channel canal. The dynamics are given by the shallow water equations [47]

$$\partial_t \begin{bmatrix} h \\ u \end{bmatrix} + \begin{bmatrix} u & h \\ g & u \end{bmatrix} \partial_z \begin{bmatrix} h \\ u \end{bmatrix} = 0, \tag{1.8}$$

with $h(z,t)$ the height of the water level, $u(z,t)$ the water speed and g the acceleration due to gravity, with z being the spatial variable representing the length of the canal i.e., $z \in [0,l]$. The first equation expresses the mass-balance and the second equation comes from the momentum-balance. The total energy (Hamiltonian) is given by

$$\mathcal{H} = \frac{1}{2}\int_0^l [hu^2 + gh^2]dz. \tag{1.9}$$

We can write this as an infinite-dimensional port-Hamiltonian system as follows: The energy variables are the height $h(z,t)$ and the velocity $u(z,t)$. The energy exchange of the system with the environment takes place through the boundary $\{0, l\}$ of the system. The dynamics (1.8) can be written in the form

$$\begin{aligned} \frac{\partial h}{\partial t} &= \partial_z(hu) = \partial_z(\delta_u \mathcal{H}) \\ \frac{\partial u}{\partial t} &= \partial_z(\frac{1}{2}u^2 + gh) = \partial_z(\delta_h \mathcal{H}). \end{aligned} \tag{1.10}$$

The boundary variables are the mass flow hu and the Bernoulli function $\frac{1}{2}u^2 + gh$ evaluated at $\{0, l\}$. The system (1.10) is an infinite-dimensional port-Hamiltonian system defined with respect to a Stokes-Dirac structure and it satisfies the energy balance

$$\frac{d\mathcal{H}}{dt} = (hu)(\frac{1}{2}u^2 + gh) \mid_0^l .$$

This means that the change in energy in the spatial domain is equal to the energy exchanged from the boundary of the system. Observe that hu times $\frac{1}{2}u^2 + gh$ has the dimensions of power. We shall elaborate on this model and its properties in Chapter 2.

1.4.1 Spatial discretization of infinite-dimensional port-Hamiltonian systems

Consider a mixed finite and infinite-dimensional port-Hamiltonian system, where we interconnect finite-dimensional systems to infinite dimensional systems. Such an interconnection defines again a port-Hamiltonian system, as will be discussed thoroughly in Chapter 3. A typical example of such a system is a power-drive consisting of a power converter, transmission line and electrical machine. From the control and simulation point of view of such systems, it may be crucial to approximate the infinite-dimensional subsystem with a finite-dimensional one. The finite-dimensional approximation should be such that it is again a port-Hamiltonian system which retains all the properties of the infinite-dimensional model, like energy balance and other conserved quantities. Furthermore, the port-variables of the approximated system should be such that it can easily be replaced in the original system, in other words the original interconnection constraints should be retained. It has been shown in [18] how the intrinsic Hamiltonian formulation suggests finite element methods which result in finite-dimensional approximations which are again port-Hamiltonian systems. Given the port-Hamiltonian formulation of distributed parameter systems it is natural to use *different* finite-elements for the approximation of functions and forms. In [18] this method was used for discretization of the ideal transmission line and the two dimensional wave equation. In Chapter 6 we apply these methods for spatial discretization of the shallow water equations.

1.5 Outline of the thesis

- In Chapter 2 we start with an introduction to the concepts of Dirac structures and port-Hamiltonian systems. We present, from the literature, some basic results on various representations of Dirac structures and port-Hamiltonian systems. We also state how the concepts of Dirac structures and port-Hamiltonian systems can be extended to infinite-dimensional systems in order the incorporate energy flow though the boundary. The port-Hamiltonian models of the shallow water equations are based on the results obtained in [46].

- Chapter 3 focuses on power-conserving interconnections of port-Hamiltonian systems by studying compositions of their Dirac structures. Since the interconnection preserves the power in the system, the resulting system is again energy-conserving. It is then shown that the resulting system can also be described as a port-Hamiltonian system. We study composition of finite-dimensional Dirac structures with finite-dimensional Dirac structures and composition of Dirac structures with resistive relations. The composition of Dirac structures with resistive relations enables us to study interconnections of various port-Hamiltonians systems with dissipation. The results are extended also to the infinite-dimensional case, where we study composition of infinite-dimensional Dirac structures with or without dissipation. This composition could either be through the spatial domain or through the boundary of the infinite-dimensional Dirac structure. Finally, we study interconnections of systems which are mixed in nature, that is composition of finite-dimensional Dirac structure with infinite-dimensional Dirac structures. In particular we study two cases, 1) Interconnections of two finite-dimensional systems via an infinite-dimensional system and 2) Interconnection of two infinite-dimensional systems via a *distributed* finite-dimensional system. The results in this chapter are based on the papers [42, 43].

- Chapter 4 uses the results on interconnections for further analysis of port-Hamiltonian systems. We investigate, which closed-loop port-Hamiltonian systems can be achieved by interconnecting a given plant port-

Hamiltonian system P with a to-be-designed controller port-Hamiltonian system C. We also characterize the set of achievable Casimirs for the closed-loop systems and study its implications on control of port-Hamiltonian systems. We also focus on the role of energy dissipation and in the case of finite-dimensional systems with dissipation we see how under certain conditions, if a function is a Casimir for a given resistive relation, it is a Casimir for all resistive relations. The results of this chapter have also been presented in [42, 43].

- In Chapter 5 we use results obtained on achievable Casimirs in Chapter 4 for control of port-Hamiltonian systems. In particular we are interested in the problem of set point regulation. We use the Casimirs in the extended state-space to generate Lyapunov functions of the closed-loop system as the sum of the plant and the controller Hamiltonians and the corresponding Casimir function. We also see, how with the help of new passive outputs we can study stability of forced port-Hamiltonian systems with dissipation.
  In the case of control of infinite-dimensional port-Hamiltonian systems, we consider the problem of stabilization (by generating Casimir functions in the extended state space) in the case where the plant system, to be controlled, consists of an infinite-dimensional subsystem. We study asymptotic stability of infinite-dimensional systems by injecting damping through the boundary of the system. We also explore the possibility of extending this control techniques to control of fluid dynamical systems. We present some preliminary results with applications to the shallow water equations. The material in this chapter is based on the papers [44, 16, 26].

- Chapter 6 deals with spatial discretization of the one dimensional shallow water equations, which are modeled as infinite-dimensional port-Hamiltonian systems defined with respect to a Stokes-Dirac structure. We use the idea presented in [18] of using mixed finite elements for discretizing different types of differential forms. We present some preliminary numerical results and also present a simple extension to the case of spatial discretization of non-constant Stokes-Dirac structure. This chapter is primarily based on the papers [41, 45].

2

Port-Hamiltonian Systems

"Equations are just the boring part of mathematics. I attempt to see things in terms of geometry." - Stephen Hawking.

In the previous chapter we have given a brief introduction on how network modeling of physical systems, both finite and infinite-dimensional, leads to port-Hamiltonian systems. Port-Hamiltonian systems are described by a geometric object called the Dirac structure which captures the interconnection structure and the physical laws of the system. A Dirac structure generalizes the notions of symplectic and Poisson structures. We also stated that this framework can be extended to model infinite-dimensional systems incorporating boundary energy flow.

In this chapter we discuss concepts in detail. To begin with, we recall the definition of a Dirac structure and show it can be used to define a port-Hamiltonian system and discuss various representations of Dirac structures and port-Hamiltonian systems. We also state the definition of an infinite-dimensional port-Hamiltonian system defined with respect to an infinite-dimensional Dirac structure called the Stokes-Dirac structure. Towards the end we present a few examples which we use in the rest of the thesis.

2.1 Port-Hamiltonian systems and Dirac structures

2.1.1 Dirac structures

To define the notion of Dirac structures for finite dimensional systems, we start with a space of power variables $\mathcal{F} \times \mathcal{F}^*$. We call $\mathcal{F}$ the space of flows whose elements are denoted by $f \in \mathcal{F}$ and are called flow vectors. The space of efforts is given by the dual linear space $\mathcal{E} := \mathcal{F}^*$, and its elements are denoted by $e \in \mathcal{E}$. The total space of flow and effort variables is $\mathcal{F} \times \mathcal{F}^*$ and will be called the space of port variables. On this space of port variables, we define the power by

$$P =< e \mid f >, \ (f, e) \in \mathcal{F} \times \mathcal{F}^*,$$

where $< e \mid f >$ denotes the duality product, that is, the linear functional $e \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$.

Definition 2.1. [58] A Dirac structure on $\mathcal{F} \times \mathcal{F}^*$ is a subspace

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*,$$

such that
(i) $< e \mid f >= 0$, for all $(f, e) \in \mathcal{D}$ and
(ii) $\dim \mathcal{D} = \dim \mathcal{F}$.

Property *(i)* corresponds to power conservation and expresses the fact that the total power entering (or leaving) the Dirac structure is zero. It can be shown that the maximal dimension of any subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ satisfying property *(i)* is equal to $\dim \mathcal{F}$. Instead of proving this directly, we will give an equivalent definition of a Dirac structure from which the claim immediately follows. Furthermore, this equivalent definition of a Dirac structure has the advantage that it generalizes to the case of an infinite-dimensional linear space $\mathcal{F}$, leading to the definition of an infinite-dimensional Dirac structure. This is instrumental in the definition of a distributed-parameter port-Hamiltonian system as will be seen in Section 2.3.

In order to give this equivalent characterization of a Dirac structure, we look more closely at the geometric structure of the total space of flow and effort variables $\mathcal{F} \times \mathcal{F}^*$. In fact, related to the definition of power, there exists a canonically defined bilinear form $\ll, \gg$ on the space $\mathcal{F} \times \mathcal{F}^*$, defined as

$$\begin{aligned} \ll (f^a, e^a), (f^b, e^b) \gg \ &:=< e^a \mid f^b > + < e^b \mid f^a >, \\ &(f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{F}^*. \end{aligned} \tag{2.1}$$

Note that this bilinear form is indefinite, that is, $\ll (f, e), (f, e) \gg$ may be positive or negative, but it is non-degenerate, that is, $\ll (f^a, e^a), (f^b, e^b) \gg= 0$ for all (f^b, e^b) implies that $(f^a, e^a) = 0$.

Proposition 2.2. *[58] A constant Dirac structure on $\mathcal{F} \times \mathcal{F}^*$ is a subspace*

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*,$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp}, \tag{2.2}$$

where $\perp$ denotes the orthogonal complement with respect to the bilinear form $\ll, \gg$.

Proof. Let $\mathcal{D}$ satisfy (2.2). Then for every $(f, e) \in \mathcal{D}$

$$\begin{aligned} 0 &=\ll (f, e), (f, e) \gg \\ &=< e \mid f > + < e \mid f > \\ &= 2 < e \mid f > . \end{aligned}$$

By non-degeneracy of $\ll, \gg$ $\dim \mathcal{D}^\perp = \dim(\mathcal{F}\times\mathcal{F}^*) - \dim \mathcal{D} = 2\dim\mathcal{F} - \dim\mathcal{D}$ and hence property (2.2) implies that $\dim \mathcal{D} = \dim \mathcal{F}$.

Conversely, let $\mathcal{D}$ be a Dirac structure and thus satisfying properties (i) and (ii) of Definition 2.1. Let $(f^a, e^a), (f^b, e^b)$ be any vectors contained in $\mathcal{D}$. Then by linearity $(f^a + f^b, e^a + e^b) \in \mathcal{D}$. Hence by property (i)

$$\begin{aligned} 0 &=< e^a + e^b \mid f^a + f^b > \\ &=< e^a \mid f^b > + < e^b \mid f^a > + < e^a \mid f^a > + < e^b \mid f^b > \\ &=< e^a \mid f^b > + < e^b \mid f^a >=\ll (f^a, e^a), (f^b, e^b) \gg, \end{aligned}$$

since by application of property (i) $< e^a \mid f^a >=< e^b \mid f^b >= 0$. This implies that

$$\mathcal{D} \subset \mathcal{D}^\perp.$$

Furthermore, by property (ii) and $\dim \mathcal{D}^\perp = 2\dim\mathcal{F} - \dim\mathcal{D}$ it follows that

$$\dim \mathcal{D} = \dim \mathcal{D}^\perp,$$

yielding $\mathcal{D} = \mathcal{D}^\perp$. □

Remark 2.3. Note that we have actually shown that property (i) implies $\mathcal{D} \subset \mathcal{D}^\perp$. Together with the fact that $\dim \mathcal{D}^\perp = \dim \mathcal{D} = 2\dim\mathcal{F} - \dim\mathcal{D}$ it implies that any subspace $\mathcal{D}$ satisfying property (i) has the property that $\dim \mathcal{D} \leq \dim \mathcal{F}$. Thus as claimed before, a Dirac structure is a linear subspace of maximal dimension satisfying property (i).

Example 2.4. Let $\mathcal{F}$ be a linear space over $\mathbb{R}$. Let $\mathcal{E}$ be given as $\mathcal{F}^*$ (the space of linear functionals on $\mathcal{F}$), with pairing $<|>$ and the duality product $< e \mid f >\in \mathbb{R}$.

1. Let $J : \mathcal{E} \to \mathcal{F}$ be a skew-symmetric map. Then the graph $J \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.

2. Let $\omega : \mathcal{F} \to \mathcal{E}$ be a skew-symmetric map. Then the graph $\omega \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.

3. Let $V \subset \mathcal{F}$ be a finite-dimensional linear subspace. Then $V \times V^{orth} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure, where $V^{orth} \subset \mathcal{E}$ is the annihilating subspace of V. The same holds if $\mathcal{F}$ is a topological vector space, $\mathcal{E}$ is the space of linear continuous functionals on $\mathcal{F}$, and V is a closed subspace of $\mathcal{F}$.

Example 2.5 (Kirchhoff's laws as Dirac structures). Consider an electrical circuit with $n-$edges where the current through the $i-$th edge is denoted by I_i and the voltage over the $i-$th edge is V_i. Collect the currents in a single column vector I (of dimension n) and the voltages in an $n-$dimensional column vector V. The following consequence of Kirchhoff's current and voltage laws is well-known. Let Kirchhoff's current laws be written in matrix form as

$$\mathcal{A}I = 0, \tag{2.3}$$

for some matrix A (with n columns). Then Kirchhoff's voltage laws can be written in the following form. All allowed vectors of voltages V in the circuit are given as

$$V = \mathcal{A}^T \lambda, \tag{2.4}$$

for any vector λ of appropriate dimension. It is immediately seen that the total space of currents and voltages allowed by Kirchhoff's current and voltage laws

$$\mathcal{D} := \{(I, V) \mid \mathcal{A}I = 0, V = \mathcal{A}^T \lambda\},$$

defines a Dirac structure. Consequently

$$(V^a)^T I^b + (V^b)^T I^a = 0,$$

for all pairs $(I^a, V^a), (I^b, V^b) \in \mathcal{D}$. In particular, by taking V^a, I^b equal to zero, we obtain

$$(V^b)^T I^a = 0,$$

for all I^a satisfying (2.3) and all V^b satisfying (2.4). This is nothing else than *Tellegen's theorem*.

Example 2.6. A Transformer can easily be seen as an example of a Dirac structure. A transformer is a two-port linking the flow and effort variables (f_1, e_1) and (f_2, e_2) by

$$\begin{aligned} f_2 &= \alpha f_1 \\ e_1 &= -\alpha e_2, \end{aligned} \tag{2.5}$$

with α being a constant, called the transformer ratio. The subspace defined by (2.5) is easily checked to be a Dirac structure.

2.1.2 Port-Hamiltonian systems

In general, a port-Hamiltonian system can be represented as in Figure 2.1. Central in the definition of a port-Hamiltonian system is the notion of a *Dirac structure*, depicted in Figure 2.1 by $\mathcal{D}$. Basic property of any Dirac structure is *power conservation*: the Dirac structure links the various port variables in such a way that the total power associated with the port-variables is zero.

The port variables entering the Dirac structure have been split in Figure 2.1 in different parts. First, there are two *internal* ports. One, denoted by $\mathcal{S}$, is corresponding to energy-storage and the other one, denoted by $\mathcal{R}$, is corresponding to internal energy-dissipation (resistive elements). Second, two

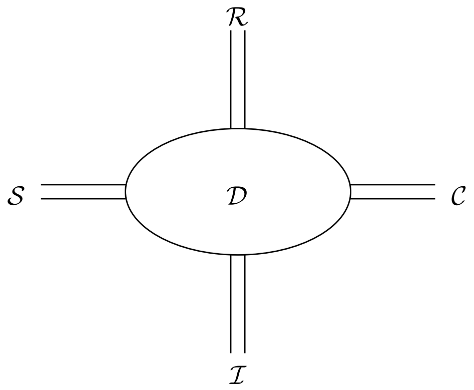


Figure 2.1: port-Hamiltonian system.

external ports are distinguished. The external port denoted by $\mathcal{C}$ is the port that is accessible for controller action. Also the presence of *sources* may be included in this port. Finally, the external port denoted by $\mathcal{I}$ is the interaction port, defining the interaction of the system with (the rest of) its environment.

Energy storage port

The port variables associated with the internal storage port will be denoted by (f_S, e_S). They are interconnected to the energy storage of the system which is defined by a finite-dimensional state space manifold $\mathcal{X}$ with coordinates x, together with a Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$ denoting the energy. The flow variables of the energy storage are given by the *rate* $\dot{x}$ of the energy variables x. Furthermore, the effort variables of the energy storage are given by the *co-energy* variables $\frac{\partial H}{\partial x}(x)$, resulting in the energy balance

$$\frac{d}{dt}H = < \frac{\partial H}{\partial x}(x) \mid \dot{x} > = \frac{\partial^T H}{\partial x}(x)\dot{x}. \tag{2.6}$$

(Here we adopt the convention that $\frac{\partial H}{\partial x}(x)$ denotes the *column* vector of partial derivatives of H.)

The interconnection of the energy storing elements to the storage port of the Dirac structure is accomplished by setting

$$\begin{array}{rcl} f_S & = & -\dot{x} \\ e_S & = & \frac{\partial H}{\partial x}(x). \end{array} \tag{2.7}$$

Hence the energy balance (2.6) can be also written as

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_S^T f_S. \tag{2.8}$$

Resistive port

The second internal port corresponds to internal energy dissipation (due to friction, resistance, etc.), and its port variables are denoted by (f_R, e_R). These port variables are terminated on a static resistive relation $\mathcal{R}$. In general, a static resistive relation will be of the form

$$R(f_R, e_R) = 0, \tag{2.9}$$

with the property that for all (f_R, e_R) satisfying (2.9)

$$< e_R \mid f_R > \leq 0. \tag{2.10}$$

In many cases we may restrict ourselves to *linear* resistive relations. (Note that some nonlinearities can be captured in the description of the resistive port of the Dirac structure.) This means that the resistive port variables (f_R, e_R) satisfy linear relations of the form

$$R_f f_R + R_e e_R = 0. \tag{2.11}$$

The inequality (2.10) corresponds to the square matrices R_f and R_e satisfying the properties of symmetry and semi-positive definiteness

$$R_f R_e^T = R_e R_f^T \geq 0, \tag{2.12}$$

together with the dimensionality condition

$$\operatorname{rank}\,[R_f | R_e] = \dim f_R. \tag{2.13}$$

Indeed, by the dimensionality condition (2.13) and the symmetry (2.12) we can equivalently rewrite the kernel representation (2.11) of $\mathcal{R}$ into an image representation

$$\begin{array}{rcl} f_R & = & R_e^T \lambda \\ e_R & = & -R_f^T \lambda. \end{array} \tag{2.14}$$

That is, any pair (f_R, e_R) satisfying (2.11) can be written into the form (2.14) for a certain λ, and conversely any (f_R, e_R) for which there exists λ such that (2.14) holds is satisfying (2.11).

Hence by (2.12) for all f_R, e_R satisfying the resistive relation

$$e_R^T f_R = -(R_f^T \lambda)^T R_e^T \lambda = -\lambda^T R_f R_e^T \lambda \leq 0. \tag{2.15}$$

Without the presence of additional external ports, the Dirac structure of the port-Hamiltonian system satisfies the power-balance

$$e_S^T f_S + e_R^T f_R = 0, \tag{2.16}$$

which leads by substitution of the equations (2.8) and (2.15) to

$$\frac{d}{dt}H = -e_S^T f_S = e_R^T f_R \leq 0. \tag{2.17}$$

An important special case of resistive relations between f_R and e_R occurs when the resistive relations can be expressed as an *input-output* mapping

$$f_R = -F(e_R), \tag{2.18}$$

where the resistive characteristic $F : \mathbb{R}^{m_r} \to \mathbb{R}^{m_r}$ satisfies

$$e_R^T F(e_R) \geq 0, \quad e_R \in \mathbb{R}^{m_r}. \tag{2.19}$$

In many cases, F will be derivable from a so-called *Rayleigh dissipation function* $R : \mathbb{R}^{m_r} \to \mathbb{R}$ in the sense that $F(e_R) = \frac{\partial R}{\partial e_R}(e_R)$.) For *linear* resistive elements, (2.18) specializes to

$$f_R = -\tilde{R} e_{R,} \tag{2.20}$$

for some positive semi-definite symmetric matrix $\tilde{R} = \tilde{R}^T \geq 0$.

Remark 2.7. The idea of using the notation in (2.10) can be explained as follows: A port-Hamiltonian system with dissipation is obtained as a composition of a Dirac structure $\mathcal{D}$ with a resistive relation $\mathcal{R}$, with $(f_S, e_S, f_R, e_R) \in \mathcal{D}$ and $(\tilde{f}_R, \tilde{e}_R) \in \mathcal{R}$. The elements in $\mathcal{R}$ are such that $< \tilde{e}_R \mid \tilde{f}_R > \geq 0$, indicating a positive power towards the element, and the resistive relation is connected to the Dirac structure via the following interconnection constraints

$$f_R = -\tilde{f}_R, \quad e_R = \tilde{e}_R.$$

Hence the flow and effort variables corresponding to the resistive ports of the Dirac structure satisfy

$$< e_R \mid f_R > = - < \tilde{e}_R \mid \tilde{f}_R > \leq 0.$$

We study the composition of Dirac structures and resistive relations in Chapter 3, and see how this composition can be used to define a port-Hamiltonian system with dissipation.

External ports

Now, let us consider in more detail the *external* ports of the system. We shall distinguish between two types of external ports. One is the *control port* $\mathcal{C}$, with port variables (f_C, e_C), which are the port variables which are accessible for controller action. The other type of external port is the *interaction port* $\mathcal{I}$, which denotes the interaction of the port-Hamiltonian system with its environment. The port variables corresponding to the interaction port are denoted by (f_I, e_I). Taking both the external ports into account the power-balance (2.16) extends to

$$e_S^T f_S + e_R^T f_R + e_C^T f_C + e_I^T f_I = 0, \tag{2.21}$$

whereby (2.17) extends to

$$\frac{d}{dt} H = e_R^T f_R + e_C^T f_C + e_I^T f_I. \tag{2.22}$$

Port-Hamiltonian dynamics

The port-Hamiltonian system with state space $\mathcal{X}$, Hamiltonian H corresponding to the energy storage port $\mathcal{S}$, resistive port $\mathcal{R}$, control port $\mathcal{C}$, interconnection port $\mathcal{I}$, and total Dirac structure $\mathcal{D}$ will be succinctly denoted by $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{C}, \mathcal{I}, \mathcal{D})$. The dynamics of the port-Hamiltonian system are specified by considering the constraints on the various port variables imposed by the Dirac structure, that is,

$$(f_S, e_S, f_R, e_R, f_C, e_C, f_I, e_I) \in \mathcal{D},$$

and to substitute in these relations the equalities $f_S = -\dot{x}, e_S = \frac{\partial H}{\partial x}(x)$. This leads to the implicitly defined dynamics

$$(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_R(t), e_R(t), f_C, (t), e_C(t), f_I(t), e_I(t)) \in \mathcal{D}, \tag{2.23}$$

with $f_R(t), e_R(t)$ satisfying for all t the resistive relation

$$R_f f_R(t) + R_e e_R(t) = 0. \tag{2.24}$$

In many cases of practical interest, the dynamics (2.23) will constrain the allowed states x, depending on the values of the external port variables (f_C, e_C) and (f_I, e_I). Thus in an equational representation (as will be treated in detail in the next section), port-Hamiltonian systems generally will consist of a mixed set of *differential* and *algebraic* equations (DAEs).

2.1.3 Input-state-output port-Hamiltonian systems

An important special case of port-Hamiltonian systems as defined above is the class of *input-state-output port-Hamiltonian systems*, where there are no algebraic constraints on the state space variables, and the flow and effort variables of the resistive, control and interaction port are split into conjugated input-output pairs. (Note that the absence of algebraic constraints on the state variables can be alternatively formulated by requiring that the effort variables at the energy-storage port, which are given as the components of the gradient of the Hamiltonian, are input variables.)

Input-state-output port-Hamiltonian systems (without feedthrough terms) are of the form

$$\Sigma: \begin{array}{rcl} \dot{x} & = & [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u + k(x)d \\ y & = & g^T(x)\frac{\partial H}{\partial x}(x) \qquad\qquad\qquad\qquad x \in \mathcal{X}, \\ z & = & k^T(x)\frac{\partial H}{\partial x}(x) \end{array} \tag{2.25}$$

where (u, y) are the input-output pairs corresponding to the control port $\mathcal{C}$, while (d, z) denote the input-output pairs of the interaction port $\mathcal{I}$. Here the matrix $J(x)$ is skew-symmetric, that is $J(x) = -J^T(x)$. The matrix $R(x) = R^T(x) \geq 0$ specifies the resistive relation. From a resistive port point of view, it is given as $R(x) = g_R^T(x)\tilde{R}g_R(x)$ for some linear resistive relation $f_R = -\tilde{R}e_R, \tilde{R} = \tilde{R}^T \geq 0$, with g_R representing the input matrix corresponding to the resistive port. The underlying Dirac structure of the system is then given by the graph of the skew-symmetric linear map

$$\begin{pmatrix} -J(x) & -g_R(x) & -g(x) & -k(x) \\ g_R^T(x) & 0 & 0 & 0 \\ g^T(x) & 0 & 0 & 0 \\ k^T(x) & 0 & 0 & 0 \end{pmatrix}. \tag{2.26}$$

Note that $y^T u$ and $z^T d$ equal the power corresponding to the control, respectively, interaction port. In general, the Dirac structure defined as the graph of the mapping (2.26) is a *modulated Dirac structure* since the matrices J, g_R, g, k may all depend on the energy variables x.

By allowing extra non-zero terms in the skew-symmetric map (2.26), a more general form of input-state-output port-Hamiltonian systems than (2.25) is obtained, including in particular feedthrough terms. For further details we refer to [14, 58].

Example 2.8 (LC-circuit with independent storage elements). Consider a controlled LC-circuit (see Figure 2.2) consisting of two inductors with magnetic energies $H_1(\varphi_1)$ and $H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), and a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear, then $H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2$, $H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2$, and $H_3(Q) = \frac{1}{2C}Q^2$. Furthermore, let $V = u$ denote a voltage source. Using Kirchhoff's laws, one immediately arrives at the input-state-output port-Hamiltonian sys-

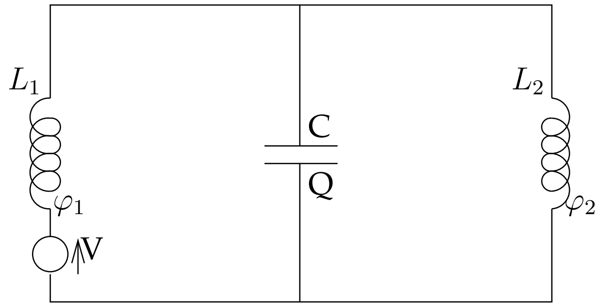


Figure 2.2: Controlled LC-circuit

tem

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{2.27}$$

$$y = \frac{\partial H}{\partial \varphi_1} \qquad \text{(= current through first inductor)},$$

with $H(Q, \varphi_1, \varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ the total energy. Clearly the matrix J is skew-symmetric.

Example 2.9. Consider the capacitor microphone as in Figure 2.3. Here the capacitance $C(q)$ is varying as a function of the displacement q of the right plate (with mass m), which is attached to a spring (with spring constant $k > 0$) and a damper (with constant $c > 0$) and affected by a mechanical force F (air pressure arising from sound). Furthermore, E is a voltage source. The dynamical equations of motion can be written as the port-Hamiltonian system with dissipation

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -c & 0 \\ 0 & 0 & -1/R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/R \end{bmatrix} \begin{bmatrix} F \\ E \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} = \dot{q} \\ \frac{1}{R}\frac{\partial H}{\partial Q} = I \end{bmatrix}.$$

with p the momentum, R the resistance of the resistor, I the current through the voltage source and the Hamiltonian given by

$$H(q, p, Q) = \frac{1}{2m} p^2 + \frac{1}{2} k (q - \tilde{q})^2 + \frac{q}{2A\epsilon} Q^2,$$

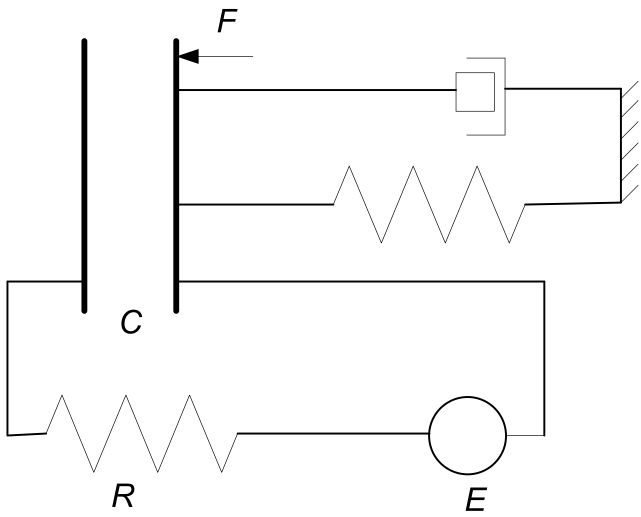


Figure 2.3: Capacitor microphone

where A is the plate area and ϵ the permittivity in the gap. Note that $F\dot{q}$ is the mechanical power and EI the electrical power applied to the system. In the application as a microphone the voltage over the resistor will be used (after amplification) as a measure for the mechanical force F.

2.2 Representations of Dirac structures and port-Hamiltonian systems

In the preceding section, we have provided the geometric definition of a port-Hamiltonian system containing three main ingredients. First, the energy storage which is represented by a state space manifold $\mathcal{X}$ specifying the space of energy variables together with a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ defining the energy. Secondly, there are the static resistive elements, and thirdly there is the Dirac structure linking all the flows and efforts associated to the energy storage, resistive elements, and the external ports (e.g. control and interaction ports) in a power-conserving manner. This leads to a definition of a port-Hamiltonian system that is *coordinate-free* because of three reasons: (a) we do not start with coordinates for the state space manifold $\mathcal{X}$, (b) we define the Dirac structure as a *subspace* instead of a set of equations, (c) also the resistive relations are defined as a subspace constraining the port variables f_R, e_R.

This geometric, coordinate-free, point of view has a number of advantages. It allows one to reason about port-Hamiltonian systems without the need to choose specific representations. In the coming chapters we will see that a number of properties of the port-Hamiltonian system, such as passivity and existence of conserved quantities, can be analyzed without the need for choosing coordinates and equations.

On the other hand, for many purposes, e.g. simulation, the need for an equational representation of the port-Hamiltonian system is indispensable. Then the emphasis shifts to finding the most convenient equational representation for the purpose at hand. In this section, we will briefly discuss a number of possible representations of port-Hamiltonian systems. It will turn out that the key issue is the representation of the Dirac structure.

2.2.1 Representations of Dirac structures

Dirac structures admit different *representations*. Here we just list a number of them; see [6, 10, 58] for more information.

1. *(Kernel and Image representation)* Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented in *kernel representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \ \times \ \mathcal{F}^* \mid Ff + Ee = 0\}, \tag{2.28}$$

for linear maps $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying

$$\begin{aligned} (i) \quad & EF^* + FE^* = 0, \\ (ii) \quad & \mathrm{rank}(F + E) = \dim \mathcal{F}, \end{aligned} \tag{2.29}$$

where $\mathcal{V}$ is a linear space with the same dimension as $\mathcal{F}$, and where $F^* : \mathcal{V}^* \to \mathcal{F}^*$ and $E^* : \mathcal{V}^* \to \mathcal{F}^{**} = \mathcal{F}$ are the adjoint maps of F and E, respectively.

It follows from (2.29) that $\mathcal{D}$ can be also written in *image representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \ \times \ \mathcal{F}^* \mid f = E^*\lambda, e = F^*\lambda, \lambda \in \mathcal{V}^*\}. \tag{2.30}$$

Sometimes it will be useful to relax this choice of the linear mappings F and E by allowing $\mathcal{V}$ to be a linear space of dimension greater than the dimension of $\mathcal{F}$. In this case we shall speak of *relaxed* kernel and image representations.

Matrix kernel and image representations are obtained by choosing linear coordinates for $\mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{V}$. Indeed, take any basis $f_1, \cdots, f_n$ for $\mathcal{F}$ and the *dual basis* $e_1 = f_1^*, \cdots, e_n = f_n^*$ for $\mathcal{F}^*$, where dim $\mathcal{F} = n$. Furthermore, take any set of linear coordinates for $\mathcal{V}$. Then the linear maps F and E are represented by $n \times n$ matrices F and E satisfying

$$\begin{aligned} (i) \quad & EF^T + FE^T = 0, \\ (ii) \quad & \text{rank}\,[F \mid E] = \dim \mathcal{F}. \end{aligned} \tag{2.31}$$

In the case of a relaxed kernel and image representation F and E will be $n' \times n$ matrices with $n' > n$.

2. (*Constrained input-output representation*) Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je + G\lambda, G^T e = 0\}, \tag{2.32}$$

for a skew-symmetric mapping $J : \mathcal{F} \to \mathcal{F}^*$ and a linear mapping G such that $\operatorname{im} G = \{f \mid (f, 0) \in \mathcal{D}\}$. Furthermore, $\ker J = \{e \mid (0, e) \in \mathcal{D}\}$.

3. *(Hybrid input-output representation)* [6]. Let $\mathcal{D}$ be given in matrix kernel representation by square matrices E and F as in (2.28). Suppose rank $F = m(\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = [F_1 \vdots F_2]$, and correspondingly $E = [E_1 \vdots E_2]$, $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. Then the matrix $[F_1 \vdots E_2]$ is invertible, and

$$\mathcal{D} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \middle| \begin{bmatrix} f_1 \\ e_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \right\}, \tag{2.33}$$

with $J := -[F_1 \vdots E_2]^{-1}[F_2 \vdots E_1]$ skew-symmetric.

4. *(Canonical coordinate representation)* [10]. There exists a basis for $\mathcal{F}$ and dual basis for $\mathcal{F}^*$, such that, in these bases, the vector (f, e), when partitioned as $(f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s)$, is in $\mathcal{D}$ if and only if

$$\left\{ \begin{array}{lllllll} f_q & = & -e_p, & f_p & = & e_q \\ f_s & = & 0, & e_s & = & 0 \end{array} \right. . \tag{2.34}$$

Remark 2.10. A special type of kernel representation occurs if not only $EF^* + FE^* = 0$ but in fact $FE^* = 0$ (while still $\operatorname{rank}(F + E) = \dim \mathcal{F}$). In this case it follows that $\ker F = \operatorname{im} E^*$, and the Dirac structure is the product of the subspace $\ker F \subset \mathcal{F}$ and the subspace $\ker F^{\text{orth}} = \ker E \subset \mathcal{F}^*$.

In [12, 6] it is shown how one may convert any of the above representations into any other. A particular transformation that will be used in the sequel is the direct transformation of the *constrained input-output* representation into the *kernel* representation. Consider the Dirac structure $\mathcal{D}$ given in constrained input-output representation by (2.32). Construct a linear mapping $G^\perp$ of maximal rank satisfying

$$G^\perp G = 0.$$

Then, premultiplying the first equation of (2.32) by $G^{\perp}$, one eliminates the Lagrange multipliers λ and obtains

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid G^{\perp} f = G^{\perp} J e, G^T e = 0\}, \tag{2.35}$$

which is easily seen to define a kernel representation.

2.2.2 Representations of port-Hamiltonian systems

Equational representations of the port-Hamiltonian system (2.23) are obtained by choosing a specific equational representation of the Dirac structure $\mathcal{D}$. For example, if $\mathcal{D}$ is given in matrix kernel representation

$$\mathcal{D} = \{(f_S, e_S, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^* \mid F_S f_S + E_S e_S + F f + E e = 0\}, \tag{2.36}$$

with

$$\begin{aligned} &\text{(i)} \quad E_S F_S^T + F_S E_S^T + E F^T + F E^T = 0, \\ &\text{(ii)} \quad \text{rank}\, [F_S \vdots E_S \vdots F \vdots E] = \dim(\mathcal{X} \times \mathcal{F}), \end{aligned} \tag{2.37}$$

then the port-Hamiltonian system is given by the set of equations

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) + F f(t) + E e(t). \tag{2.38}$$

Note that, in general, (2.38) consists of differential equations *and* algebraic equations in the state variables x (DAEs), together with equations relating the state variables and their time-derivatives to the external port variables (f, e).

Passivity of port-Hamiltonian systems

By the power-conserving property of a Dirac structure it follows that any implicit port-Hamiltonian system satisfies the energy balance

$$\frac{dH}{dt}(x(t)) = < \frac{\partial H}{\partial x}(x(t) \mid \dot{x}(t) >= e_c^T f_c + e_I^T f_I.$$

This implies that any port-Hamiltonian system is passive with respect to the supply rate $e_c^T f_c + e_I^T f_I$ and storage function H if H qualifies as a storage function, that is H is semi-positive definite ($H \geq 0$ for all x).

2.3 Infinite-dimensional port-Hamiltonian systems

In order to extend the Hamiltonian formulation to infinite-dimensional systems, for example as in [34], a fundamental difficulty arises in the treatment of boundary conditions. The treatment of infinite-dimensional Hamiltonian systems in the literature seems mostly focused on systems with infinite spatial domain, where the variables go to zero for spatial variables tending to infinity, or on systems with boundary conditions such that the energy exchange through the boundary is zero. On the other hand, from a control and interconnection point of view it is essential to be able to describe an infinite-dimensional system with varying boundary conditions including energy exchange through the boundary, since in many applications, interaction with the environment takes place through the boundary of the system. Clear examples are telegraphers equations (describing the dynamics of a transmission line), where the boundary of the system is described by the behavior of the voltages and currents at both ends of the transmission line. In such examples it is obvious that in general the boundary exchange of power will be non-zero and that in fact one would like to consider the voltages and currents as additional boundary variables of the system, which can be interconnected to other systems.

From a mathematical point of view, it is not obvious how to incorporate non-zero energy flow through the boundary, using, for example the Poisson framework for Hamiltonian systems. For example, the Korteweg-de Vries equation a Poisson bracket can be formulated (for zero boundary conditions) with the use of the differential operator d/dz as follows:
Consider the Korteweg-de Vries equation:

$$u_t = u_{zzz} + uu_z.$$

Then

$$u_t = \frac{d}{dz}(u_{zz} + \frac{1}{2}u^2) = \mathfrak{D}\delta\mathcal{H},$$

where $\mathfrak{D} = \frac{d}{dz}$ and

$$\mathcal{H}(u) = \int [-\frac{1}{2}u_z^2 + \frac{1}{6}u^3]dz.$$

$\mathfrak{D}$ is skew-symmetric, i.e. $\mathfrak{D} = -\mathfrak{D}^*$ for zero boundary conditions, and hence it can be shown that it defines a Poisson bracket, see [34] for details.

However for boundary conditions corresponding to non-zero energy flow the differential operator is not skew-symmetric anymore. Indeed, by simple integration by parts

$$\int_Z g\frac{\partial f}{\partial z}dz = -\int_Z f\frac{\partial g}{\partial z}dz + fg\mid_Z .$$

Hence $\frac{\mathrm{d}}{\mathrm{d}z}$ is a skew-symmetric operator only if either of the following conditions are true

1. The spatial domain is infinite, and the product fg can be considered to be zero at infinity.

2. The spatial domain is finite and the product fg is zero on the boundary.

Hence the Poisson framework cannot be directly used to model systems with non-zero boundary conditions

To overcome this problem a framework has been provided in [61], by introducing a special type of infinite-dimensional Dirac structure, based on the Stokes' theorem. We briefly highlight those results here with a few examples.

Let Z be an $n-$dimensional smooth manifold with a smooth $(n-1)$ dimensional boundary ∂Z, representing the space of spatial variables. Denote by $\Omega^k(Z)$, $k = 0,1,...,n$, the space of exterior $k-$forms on Z, and by $\Omega^k(\partial Z)$, $k = 0,1,...,n-1$, the space of $k-$forms on ∂Z. (Note that $\Omega^0(Z)$ respectively $\Omega^0(\partial Z)$, is the space of smooth functions on Z, respectively ∂Z.) Clearly, $\Omega^k(Z)$ and $\Omega^k(\partial Z)$ are (infinite-dimensional) linear spaces (over $\mathbb{R}$). Furthermore there is a natural pairing between $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ given by

$$< \beta \mid \alpha >:= \int_Z \beta \wedge \alpha, \quad (\in \mathbb{R}), \tag{2.39}$$

with $\alpha \in \Omega^k(Z)$, $\beta \in \Omega^{n-k}(Z)$, where $\wedge$ is the usual wedge product of differential forms yielding the $n-$form $\beta \wedge \alpha$. In fact, the pairing (2.39) is *non-degenerate* in the sense that if $< \beta \mid \alpha >= 0$ for all α, respectively, for all β, then $\beta = 0$, respectively, $\alpha = 0$.

Similarly, there is a pairing between $\Omega^k(\partial Z)$ and $\Omega^{n-1-k}(\partial Z)$ given by

$$< \beta \mid \alpha >:= \int_{\partial Z} \beta \wedge \alpha, \quad (\in \mathbb{R}), \tag{2.40}$$

with $\alpha \in \Omega^k(Z)$, $\beta \in \Omega^{n-1-k}(Z)$. Now let us define the linear space

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z), \tag{2.41}$$

for any pair p, q of positive integers satisfying

$$p + q = n + 1, \tag{2.42}$$

and correspondingly let us define

$$\mathcal{E}_{p,q} := \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z). \tag{2.43}$$

Then the pairing (2.39) and (2.40) yields a (non-degenerate) pairing between $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$. As in the finite-dimensional case, symmetrization of this pairing yields the following bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ with values in $\mathbb{R}$:

$$\begin{aligned} &\ll (f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1), (f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \gg \\ &:= \textstyle\int_Z [e_p^1 \wedge f_p^2 + e_p^2 \wedge f_p^a + e_q^1 \wedge f_q^2 + e_q^2 \wedge f_q^1] + \int_{\partial Z} [e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1], \end{aligned} \tag{2.44}$$

where for $i = 1, 2$

$$\begin{aligned} &f_p^i \in \Omega^p(Z), && f_q^i \in \Omega^q(Z) \\ &e_p^i \in \Omega^{n-p}(Z), && e_q^i \in \Omega^{n-q}(Z) \\ &f_b^i \in \Omega^{n-p}(\partial Z), && e_b^i \in \Omega^{n-q}(\partial Z). \end{aligned} \tag{2.45}$$

The spaces of differential forms $\Omega^p(Z)$ and $\Omega^q(Z)$ will represent the energy variables of the two different physical energy domains interacting with each other, while $\Omega^{n-p}(\partial Z)$ and $\Omega^{n-q}(\partial Z)$ will denote the boundary variables whose (wedge) product represents the boundary energy flow. On $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ we define the infinite-dimensional Stokes-Dirac structure

Theorem 2.11. *[61] Consider $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ given in Equations (2.41) and (2.43) with p, q satisfying (2.42), and the bilinear form $\ll \cdot \gg$ given by (2.44). Define the following linear subspace $\mathcal{D}$ of $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$*

$$\begin{aligned} \mathcal{D} = \quad & \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}; \right. \\ & \left. \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p \mid_{\partial Z} \\ e_q \mid_{\partial Z} \end{bmatrix} \right\} \end{aligned} \tag{2.46}$$

where $|_{\partial Z}$ denotes the restriction to the boundary ∂Z, and $r = pq+1$. Then $\mathcal{D} = \mathcal{D}^\perp$, that is, $\mathcal{D}$ is a Dirac structure.

Remark 2.12. The relation (2.42) comes from the definition of the Stokes-Dirac structure (2.46) and can be explained as follows: In the definition of the Stokes-Dirac structure we equate f_p which is a $p-$form on Z to $\mathrm{d}(-1)^r e_q$ which is an $n-q+1$ form on Z. Similarly we equate f_q which is a $q-$form on Z to an $n-p+1$ form on Z. Also observe that the wedge product $e_p \mid_{\partial Z} \wedge e_q \mid_{\partial Z}$ is a $2n-p-q$ form on ∂Z which equals the dimension of the boundary i.e. $n-1$.

2.3.1 Infinite-dimensional port-Hamiltonian systems

The definition of an infinite-dimensional Hamiltonian system with respect to a Stokes-Dirac structure can now be stated as follows. Let Z be an n-dimensional manifold with boundary ∂Z, and let $\mathcal{D}$ be a Stokes-Dirac structure as in (2.46). Consider furthermore a *Hamiltonian density* (energy per volume element)

$$H : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \Omega^n(Z),$$

resulting in the total energy

$$\mathcal{H} := \int_Z H \in \mathbb{R}.$$

Recall (Equation (2.39)), that there exists a non-degenerate pairing between $\Omega^p(Z)$ and $\Omega^{n-p}(Z)$, respectively between $\Omega^q(Z)$ and $\Omega^{n-q}(Z)$. This means that $\Omega^{n-p}(Z)$ and $\Omega^{n-q}(Z)$ can be regarded as dual spaces to $\Omega^p(Z)$, respectively $\Omega^q(Z)$ (although strictly contained in their functional analytic duals). Let now $\alpha_p, \partial\alpha_p \in \Omega^p(Z)$, $\alpha_q, \partial\alpha_q \in \Omega^q(Z)$. Then under weak smoothness conditions on H

$$\begin{aligned}\mathcal{H}(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) &= \int_Z H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q, z) \\ &= \int_Z H(\alpha_p, \alpha_q, z) + \int_Z [\delta_p\mathcal{H} \wedge \partial\alpha_p + \delta_q\mathcal{H} \wedge \partial\alpha_q] \\ &\quad + \text{higher order terms in } \partial\alpha_p,\ \partial\alpha_q,\end{aligned} \tag{2.47}$$

for certain differential forms

$$\begin{aligned}&\delta_p\mathcal{H} \in \Omega^{n-p}(Z) \\ &\delta_q\mathcal{H} \in \Omega^{n-q}(Z)\end{aligned} \tag{2.48}$$

Furthermore, from the non-degeneracy of the pairing $\Omega^p(Z)$ and $\Omega^{n-p}(Z)$ respectively between $\Omega^q(Z)$ and $\Omega^{n-q}(Z)$, it immediately follows that these differential forms are uniquely determined. This means that $(\delta_p\mathcal{H}, \delta_q\mathcal{H}) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z)$ can be regarded as the (partial) variational derivatives of $\mathcal{H}$ at $(\alpha_p, \alpha_q) \in \Omega^p(Z) \times \Omega^q(Z)$.

Now consider the time function

$$(\alpha_p(t), \alpha_q(t)) \in \Omega^p(Z) \times \Omega^q(Z), \quad t \in \mathbb{R},$$

and the Hamiltonian $\mathcal{H}(\alpha_p(t), \alpha_q(t))$ evaluated along this trajectory. It follows that at any time t

$$\frac{d\mathcal{H}}{dt} = \int_Z [\delta_p\mathcal{H} \wedge \frac{\partial\alpha_p}{\partial t} + \delta_q\mathcal{H} \wedge \frac{\partial\alpha_q}{\partial t}]. \tag{2.49}$$

The differential forms $\frac{\partial\alpha_p}{\partial t}$ and $\frac{\partial\alpha_q}{\partial t}$ represent the generalized velocities of the energy variables α_p, α_q. They are connected to the Stokes-Dirac structure by setting

$$\begin{aligned}f_p &= -\tfrac{\partial\alpha_p}{\partial t} \\ f_q &= -\tfrac{\partial\alpha_q}{\partial t},\end{aligned} \tag{2.50}$$

(again the minus sign is included to have a consistent energy flow description). Since the right-hand side of Equation (2.49) is the rate of increase of the stored energy $\mathcal{H}$, we set

$$\begin{aligned}e_p &= \delta_p\mathcal{H} \\ e_q &= \delta_q\mathcal{H}.\end{aligned} \tag{2.51}$$

Now we come to the general Hamiltonian description of an infinite-dimensional system with boundary energy flow.

Definition 2.13. The distributed-parameter port-Hamiltonian system with an $n-$dimensional manifold of spatial variables Z, state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p+q = n+1$), Stokes-Dirac structure $\mathcal{D}$ given by Equation (2.46) and Hamiltonian $\mathcal{H}$, is given as (with $r = pq+1$)

$$\begin{bmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_p \mathcal{H} \\ \delta_q \mathcal{H} \end{bmatrix};$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \delta_p \mathcal{H} \mid_{\partial Z} \\ \delta_q \mathcal{H} \mid_{\partial Z} \end{bmatrix}. \tag{2.52}$$

By the power-conserving property of Stokes-Dirac structure it immediately follows that for any $(f_p, f_q, f_b, e_p, e_q, e_b)$ in the Stokes-Dirac structure $\mathcal{D}$

$$\int_Z [e_p \wedge f_p + e_q \wedge f_q] + \int_{\partial Z} e_b \wedge f_b = 0.$$

Hence, by substitution of Equations (2.50) and (2.51) and using (2.49) we obtain

Proposition 2.14. *[61] Consider the distributed-parameter port-Hamiltonian system Equation (2.52). Then*

$$\frac{d\mathcal{H}}{dt} = \int_{\partial Z} e_b \wedge f_b, \tag{2.53}$$

expressing that the increase in energy on the domain Z is equal to the power supplied to the system though the boundary ∂Z.

Notations

In this thesis we mostly deal with infinite-dimensional systems with a 1-D spatial domain, which means that we distinguish between zero-forms and one-forms defined on the spatial domain of the system. One forms are objects which can be integrated over every sub-interval of the interval where as zero-forms or functions can be evaluated at any points of the interval. If we consider a spatial coordinate z for the interval Z, then a function is simply given by the values $f(z) \in \mathbb{R}$ for every coordinate value in z in the interval, while a one-form g is given as $\tilde{g}(z)dz$ for a certain density function g. We denote the set of zero forms and one-forms on Z by $\Omega^0(Z)$ and $\Omega^1(Z)$ respectively. Given a coordinate z for the spatial domain we obtain by spatial differentiation of a function $f(z)$ the one-form $\omega := \frac{\mathrm{d}f}{dz}(z)dz$. In coordinate free language this is denoted as $\omega = \mathrm{d}f$, where d is called the exterior derivative mapping zero forms to one forms.

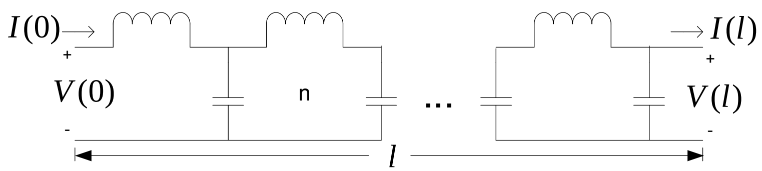


Figure 2.4: Transmission line

In case of a two-dimensional spatial domain Z (as in the case of the 2-D shallow water equations, which will be discussed in Chapter 3) we have to distinguish between zero-forms (functions), one-forms and two-forms. Again, functions are objects which can be evaluated at any point in the spatial domain. One-forms are objects which can be integrated along any line segment in the spatial domain, while two-forms are objects which can be integrated over any part of the spatial domain. Given coordinates z_1, z_2 for the spatial domain, a function is simply given by the values $f(z_1, z_2) \in \mathbb{R}$ for every point (z_1, z_2), while a one-form is expressed as $g_1(z_1, z_2)dz_1 + g_2(z_1, z_2)dz_2$ for certain functions g_1, g_2. Finally a two-form $\omega \in \Omega^2(Z)$ is given by the infinitesimal area element $k(z_1, z_2)dz_1dz_2$ for a certain function k. By spatial differentiation of a function $f(z_1, z_2)$ we obtain the one-form $\partial f/\partial z_1(z_1, z_2)dz_1 + \partial f/\partial z_2(z_1, z_2)dz_2$, while spatial differentiation of a one-form $g_1(z_1, z_2)dz_1 + g_2(z_1, z_2)dz$ results in the two-form $(\partial g_2/\partial z_1(z_1, z_2) - \partial g_1/\partial z_2(z_1, z_2))dz_1dz_2$.

Furthermore, given a k-form ω_1 and an l-form ω_2, the wedge product $\omega_1 \wedge \omega_2$ is a $k + l$-form. Finally, we will use the Hodge star operator $*$, converting any $k-$form ω on a n-dimensional spatial domain Z to an $(n - k)-$form $*\omega$. The definition of the Hodge star operator relies on the assumption of a Riemannian metric on the spatial domain Z; however, on our context this Riemannian metric will simply be the Euclidean inner product corresponding to a choice of local coordinates on Z. Thus on the one-dimensional spatial domain Z we simply have $*g(z) = \tilde{g}(z)$. or in other words $*(\tilde{g}(z)dz) = \tilde{g}(z)$.

Example 2.15 (The Ideal transmission line). In this example, we first derive the model of a transmission line, whose dynamics are described by the well-known telegrapher's equations, using the classical technique and then in the port-Hamiltonian framework. In the course of deriving the distributed parameter representation of a dynamical system, it is typical to start with spatially distributed finite lumps and then take the limit as the lumps become infinitesimal in size. Consider the transmission line of Figure 2.4, the dynamics of the n-th mesh are given by the following set of equations (also see

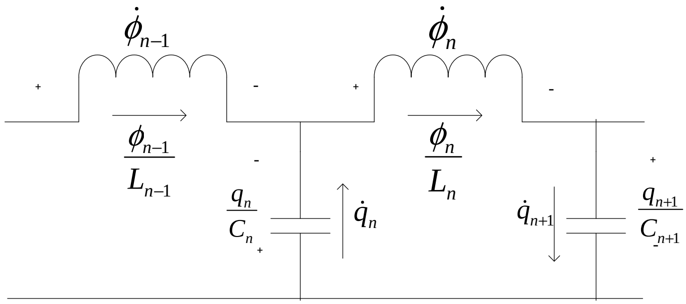


Figure 2.5: $n-$th Lumped element of the Transmission line

Figure 2.5)

$$\begin{aligned} \dot{q}_n = &\quad \tfrac{\phi_n}{L_n} - \tfrac{\phi_{n-1}}{L_{n-1}} \\ \dot{\phi}_n = &\quad \tfrac{q_{n+1}}{C_{n+1}} - \tfrac{q_n}{C_n}. \end{aligned} \tag{2.54}$$

Here q_n denotes the charge and ϕ_n the flux linkages. Note that the first equation of (2.54) can be written as

$$\dot{q}_n = -\frac{L_n(\phi_{n-1} - \phi_n) - \phi_n(L_{n-1} - L_n)}{L_n L_{n-1}},$$

which is just an approximation of the partial derivative with respect to z of $\frac{\tilde{\phi}(z,t)}{L(z)}$, with z the spatial variable ranging in $[0, l]$, $\tilde{\phi}(z, t)$ the flux distribution and the $L(z)$ the distributed inductance. That is

$$\frac{\partial}{\partial z}\frac{\tilde{\phi}(z,t)}{L(z)} \approx \frac{L_n(\phi_{n-1} - \phi_n) - \phi_n(L_{n-1} - L_n)}{L_n L_{n-1}}\frac{1}{\delta z}.$$

Therefore in the limit of small spacing between the LC circuits, the system of ordinary differential equations reduces to a single partial differential equation

$$\frac{\partial}{\partial t}\tilde{q}(z,t) = -\frac{\partial}{\partial z}\frac{\tilde{\phi}(z.t)}{L(z)}.$$

In a similar way we obtain for the second equation of (2.54)

$$\frac{\partial}{\partial t}\tilde{\phi}(z,t) = -\frac{\partial}{\partial z}\frac{\tilde{q}(z.t)}{C(z)},$$

where $\tilde{q}(z,t)$ the charge distribution and $C(z)$ is the distributed capacitance. In the Dirac framework the model is given as follows. The spatial domain is represented by a 1-D manifold with point boundaries. The energy variables are electric charge and the magnetic flux densities, $\alpha_p = q(z,t) = \tilde{q}(z,t)dz$, $\alpha_q = \phi(z,t) = \tilde{\phi}(z,t)dz$ respectively, which are both one-forms on Z. The total energy functional becomes

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\int_0^l \left(\frac{\tilde{q}^2(z,t)}{C(z)} + \frac{\tilde{\phi}^2(z,t)}{L(z)}\right) dz \\ &= \frac{1}{2}\int_0^l \left(\frac{*q(z,t)}{C(z)}q(z,t) + \frac{*\phi(z,t)}{L(z)}\phi(z,t)\right)\end{aligned}$$

with variational derivatives given by $\delta\mathcal{H} = \begin{bmatrix}\frac{\tilde{q}(z,t)}{C(z)} & \frac{\tilde{\phi}(z,t)}{L(z)}\end{bmatrix}^T$. Power flows though the system though the boundaries $\{0,l\}$, with the boundary variables being the current and voltages at each end of the line. Then, the telegraphers equations may be expressed as a distributed-parameter port-Hamiltonian system as

$$\begin{bmatrix}\frac{\partial}{\partial t}q(z,t) \\ \frac{\partial}{\partial t}\phi(z,t)\end{bmatrix} = \begin{bmatrix}0 & \mathrm{d} \\ \mathrm{d} & 0\end{bmatrix}\begin{bmatrix}\frac{*q(z.t)}{C(z)} \\ \frac{*\phi(z.t)}{L(z)}\end{bmatrix}; \begin{bmatrix}f_b \\ e_b\end{bmatrix} = \begin{bmatrix}-\frac{*\phi(z.t)}{L(z)}|\partial Z \\ \frac{*q(z.t)}{C(z)}|\partial Z\end{bmatrix}, \tag{2.55}$$

the right hand relation of (2.55) defines voltages and currents at the boundary point $\{0,l\}$. The corresponding energy balance equation is

$$\frac{d\mathcal{H}}{dt} = \frac{*q(0,t)}{C(0)}\frac{*\phi(0,t)}{L(0)} - \frac{*q(l,t)}{C(l)}\frac{*\phi(l,t)}{L(l)}.$$

Example 2.16. [55] Consider the $n-$ dimensional wave equation

$$\mu\ddot{\omega} + E\Delta\omega = 0,\ \ \omega(z,t) \in \mathbb{R},\ z = (z_1, ..., z_n) \in Z, \tag{2.56}$$

where μ is the mass density and E is the Youngs modulus. In the Dirac framework, this model is given as follows. The spatial domain is represented by an $n-$dimensional smooth manifold with a smooth $n-1$ dimensional boundary. The energy variables are the $n-$form kinetic momentum $\rho(t, z_1, ..., z_n)$, and the $1-$form elastic strain $\epsilon(t, z_1, ..., z_n)$ $(= \frac{\partial\omega}{\partial z_1}dz_1 + ... + \frac{\partial\omega}{\partial z_n}dz_n)$. The co-energy variables are then, the $0-$form velocity

$$v(t, z_1, ..., z_n) = \frac{\partial H}{\partial \rho},$$

and the $(n-1)-$ form stress

$$\sigma(t, z_1, ..., z_n) = \frac{\partial H}{\partial \epsilon},$$

where H is the Hamiltonian density defined as

$$H(\rho, \epsilon) = \frac{1}{2}[\epsilon \wedge \sigma + \rho \wedge v].$$

The $n-$dimensional wave equation (2.56) can be written as an infinite-dimensional port-Hamiltonian system as

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \epsilon} \\ \frac{\partial H}{\partial \rho} \end{bmatrix}, \quad \begin{bmatrix} v_b \\ \sigma_b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \epsilon} |_{\partial Z} \\ \frac{\partial H}{\partial \rho} |_{\partial Z} \end{bmatrix}. \tag{2.57}$$

The co-energy variables σ and v are related to the energy variables by the constitutive relations

$$\begin{aligned} \sigma &= E * \epsilon \\ v &= \tfrac{1}{\mu} * \rho, \end{aligned} \tag{2.58}$$

where E and μ are the Youngs modulus and the mass density and $*$ denotes the Hodge star operator corresponding to a choice of a Riemannian metric on Z. Substituting $\epsilon = \mathrm{d}\omega$ in (2.58) we obtain

$$d(\dot{\omega} - \frac{1}{\mu} * \rho) = 0 \Longrightarrow \dot{\omega} = \frac{1}{\mu} * \rho + f(t).$$

Set $f(t) = 0$. Next we write the second part of (2.58) as $*\dot{\rho} = - * \mathrm{d}(E * \epsilon)$ and substitute $*\rho = \mu\dot{\omega}$. This yields (due to $\epsilon = \mathrm{d}\omega$)

$$\mu\ddot{\omega} + E(*\mathrm{d} * \mathrm{d})\omega = 0. \tag{2.59}$$

The codifferentiation $\delta : \Omega^{n-k}(Z) \to \Omega^{n-k-1}(Z)$ is a map from the space of $(n-k)-$ forms to the space of $(n-k-1)-$forms and is defined as $\delta = - * d*$, hence

$$(*\mathrm{d} * \mathrm{d})\omega = -(\delta \circ \mathrm{d})\omega = -(\delta \circ \mathrm{d} + \mathrm{d} \circ \delta)\omega = \Delta\omega, \text{ since } \delta\omega = 0.$$

The system (2.57) satisfies the energy balance law

$$\frac{d\mathcal{H}(\rho, \epsilon)}{dt} = \int_{\partial Z} \sigma_b \wedge v_b.$$

where $\mathcal{H}$ is the Hamiltonian defined as $\mathcal{H}(\rho, \epsilon) = \int_Z H(\rho, \epsilon)$.

2.3.2 The shallow water equations

The dynamics of an open-channel canal can be described by the shallow water equations given by the following set of equations [47]

$$\partial_t \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix} + \begin{bmatrix} \tilde{u} & \tilde{h} \\ g & \tilde{u} \end{bmatrix} \partial_z \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix} = 0, \tag{2.60}$$

with $\tilde{h}(z,t)$ the height of the water level, $\tilde{u}(z,t)$ the water speed and g the acceleration due to gravity, with z being the spatial variable representing the length of the canal i.e., $z \in [0,l]$. The first equation expresses the mass balance and the second equation comes from the momentum balance. The total energy (Hamiltonian) is given by

$$\mathcal{H} = \frac{1}{2}\int_0^l [\tilde{h}\tilde{u}^2 + g\tilde{h}^2]dz. \tag{2.61}$$

In case of shallow water equations the energy variables are the height $\tilde{h}(z,t)$ and the velocity $\tilde{u}(z,t)$. The energy exchange of the system with the environment takes place through the boundary $\{0,l\}$ of the system.

The Stokes-Dirac structure corresponding to the 1-D fluid flow modeled by the shallow water equations is defined as follows: The spatial domain Z is represented by a 1-D manifold with point boundaries. The height of the water flow (representing the mass density) through the canal $h(z,t) = \tilde{h}(z,t)dz$ is identified with a 1-form on Z. Note that the integral of h over a subinterval denotes the total amount of water contained in that subinterval. Furthermore, assuming the existence of a *Riemannian metric* $<,>$ on Z, we identify (by index raising w.r.t this Riemannian metric) the Eulerian vector field u on Z with a 1-form. This leads to the consideration of the (linear) space of energy variables

$$X := \Omega^1(Z) \times \Omega^1(Z).$$

To identify the boundary variables we consider space of 0-forms, i.e., space of functions on ∂Z, to represent both the boundary flow and the dynamic pressure at the boundary. We thus consider the space of boundary variables

$$\Omega^0(\partial Z) \times \Omega^0(\partial Z).$$

Proposition 2.17. *Let $Z \subset \mathbb{R}$ be a 1-dimensional manifold with boundary ∂Z. Consider $V = \Omega^1(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z)$ and $V^* = \Omega^0(Z) \times \Omega^0(Z) \times \Omega^0(\partial Z)$, together with the bilinear form*

$$\begin{aligned} &<< (f_h^1, f_u^1, f_b^1, e_h^1, e_u^1, e_b^1), (f_h^2, f_u^2, f_b^2, e_h^2, e_u^2, e_b^2) >> \\ &:= \int_Z (e_h^1 \wedge f_h^2 + e_h^2 \wedge f_h^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1) \\ &+ \int_{\partial Z} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1). \end{aligned} \tag{2.62}$$

with $f_h^i, f_v^i \in \Omega^1(Z)$, $e_h^i, e_v^i, f_b^i, e_b^i \in \Omega^0(\partial Z)$
Then, $\mathcal{D} \subset V \times V^*$ defined as

$$\begin{aligned} \mathcal{D} = \;& \{(f_h, f_u, f_b, e_h, e_u, e_b) \in V \times V^* \\ & f_h = \mathrm{d}e_u, f_u = \mathrm{d}e_h, f_b = e_h \mid_{\partial Z}, e_b = -e_u \mid_{\partial Z}\}. \end{aligned} \tag{2.63}$$

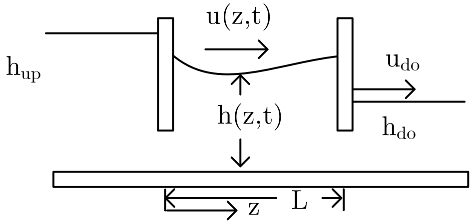


Figure 2.6: Flow of water through a canal.

where d *is the exterior derivative (mapping* $0-$*forms into* 1- *forms),* $|_{\partial W}$ *denoting the restriction of* 0*-forms on* Z *to* 0*-forms on the boundary* ∂Z*, is a Dirac structure with respect to the bilinear form* $<<,>>$ *defined as above, that is* $\mathcal{D} = \mathcal{D}^{\perp}$*, where* $\perp$ *is with respect to (2.62).* $\mathcal{D}$ *is called a Stokes-Dirac structure. In terms of shallow water equations the above terms would correspond to*

$$
\begin{aligned}
f_h &= -\frac{\partial}{\partial t}h(z,t), e_h = \delta_h \mathcal{H} = \frac{1}{2}(*u)(*u) + g * h \\
f_u &= -\frac{\partial}{\partial t}u(z,t), e_u = \delta_u \mathcal{H} = *h * u \\
f_b &= \delta_u \mathcal{H} \mid_{\partial W}, e_b = -\delta_h \mathcal{H} \mid_{\partial W},
\end{aligned}
\tag{2.64}
$$

with the Hamiltonian given as

$$
\mathcal{H} = \int_Z \frac{1}{2}(*u)h(*u) + \frac{1}{2}g(*h)h.
$$

Substituting (2.64) into (2.63), we obtain the shallow water equations (2.60).

Proof. The proof follows the same arguments as in [61], making use of the Stokes' theorem and hence we omit the proof here. □

Energy Balance

Energy balance follows immediately from the power-conserving property of the Stokes-Dirac structure, given by

$$
\int_Z (e_h \wedge f_h + e_u \wedge h_u) + \int_{\partial Z} e_b \wedge f_b = 0,
$$

and hence

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \int_{\partial Z} e_b f_b \\
&= \tilde{h}\tilde{u}(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \mid_0^L \\
&= (\tilde{u}(\frac{1}{2}\tilde{h}\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2)) \mid_0^L + (\tilde{u}(\frac{1}{2}g\tilde{h}^2)) \mid_0^L .
\end{aligned}
$$

The first term in last line of the above expression for energy balance corresponds to the energy flux (the total energy times the velocity) through the boundary and the second term is the work done by the hydrostatic pressure given by pressure times the velocity.

2.3.3 Example of a non-constant Stokes-Dirac structure

We consider a slightly different and more complicated case in which we consider an additional component of the velocity $v(z,t)$ as shown in the Figure (2.7). In addition, we assume that the height h, the horizontal velocity u and the additional velocity component v do not depend on this additional coordinate and hence we can still model this as a 1-D fluid flow as shown below. The dynamics of the system are described by the following set of equations [47]

$$
\begin{aligned}
\partial_t \tilde{h} &= -\partial_z(\tilde{h}\tilde{u}) \\
\partial_t(\tilde{h}\tilde{u}) &= -\partial_z(\tilde{h}\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2) \\
\partial_t(\tilde{h}\tilde{v}) &= -\partial_z(\tilde{h}\tilde{u}\tilde{v}),
\end{aligned} \tag{2.65}
$$

with $\tilde{h}(z,t)$ the height of the water level, $\tilde{u}(z,t)$ the water velocity in the z direction and $\tilde{v}(z,t)$ the additional component of the velocity with g the acceleration due to gravity. The first equation again corresponds to mass balance, while the second and third equations correspond to the momentum balance. The above set of equations can alternatively be written as

$$
\begin{aligned}
\partial_t \tilde{h} &= -\partial_z(\tilde{h}\tilde{u}) \\
\partial_t \tilde{u} &= -\partial_z(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \\
\partial_t \tilde{v} &= -\tilde{u}\partial_z \tilde{v}.
\end{aligned} \tag{2.66}
$$

In the port-Hamiltonian framework this is modeled as follows. The energy variables now are $h(z,t)$, $u(z,t)$ and $v(z,t)$, the Hamiltonian of the system is given by

$$
\mathcal{H} = \int_Z \frac{1}{2}\left(\tilde{h}(\tilde{u}^2 + \tilde{v}^2) + g\tilde{h}^2\right) dz, \tag{2.67}
$$

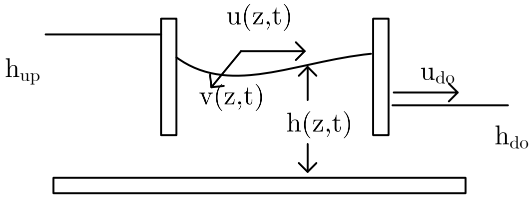


Figure 2.7: Flow with the additional velocity component.

and the variational derivatives are given by $\delta\mathcal{H} = [\frac{1}{2}(\tilde{u}^2 + \tilde{v}^2)\ \ \tilde{h}\tilde{u}\ \ \tilde{h}\tilde{v}]^T$. As before the interaction of the system with the environment takes place through the boundary of the system $\{0, l\}$. The Stokes-Dirac structure corresponding to the shallow water equations with an additional velocity component, and modeled as a 1-D fluid flow, is defined as follows: The spatial domain $Z \subset \mathbb{R}$ as before is represented by a 1-D manifold with point boundaries. The height of the water flow through the canal $h(z,t)$ is identified with a 1-form on Z and again assuming the existence of a *Riemannian metric* $<,>$ on W, we can identify (by index raising w.r.t this Riemannian metric) the Eulerian vector fields u and v on Z with a 1-form. This leads to the consideration of the (linear) space of energy variables.

$$X := \Omega^1(Z) \times \Omega^1(Z) \times \Omega^1(Z).$$

To identify the boundary variables we consider space of 0-forms, i.e., the space of functions on ∂Z, to represent the boundary height ,the dynamic pressure and the additional velocity component at the boundary. We thus consider the space of boundary variables

$$\Omega^0(\partial Z) \times \Omega^0(\partial Z) \times \Omega^0(\partial Z).$$

We will now define the Stokes-Dirac structure on $X \times \Omega^0(\partial Z)$, (i.e., the space of energy variables and part of the space of the boundary variables) in the following way

Proposition 2.18. *(Modified Stokes-Dirac structure) Let $Z \subset \mathbb{R}$ be a 1-dimensional manifold with boundary ∂Z. Consider $V = X \times \Omega^0(\partial Z) = \Omega^1(Z) \times \Omega^1(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z)$, together with the bilinear form*

$$\begin{aligned}
&<< (f_h^1, f_u^1, f_v^1, f_b^1, e_h^1, e_u^1, e_v^1, e_b^1), (f_h^2, f_u^2, f_v^2, f_b^2, e_h^2, e_u^2, e_v^2, e_b^2) >> \\
&:= \int_Z (e_h^1 \wedge f_h^2 + e_h^2 \wedge f_h^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1 + e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1) \\
&+ \int_{\partial Z} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1),
\end{aligned} \tag{2.68}$$

where

$$f_h^i \in \Omega^1(Z), f_u^i \in \Omega^1(Z), f_v^i \in \Omega^1(Z), f_b^i \in \Omega^0(\partial Z)$$
$$e_h^i \in \Omega^0(Z), e_u^i \in \Omega^0(Z), e_v^i \in \Omega^0(Z), e_b^i \in \Omega^0(\partial Z).$$

Then $\mathcal{D} \subset V \times V^*$ defined as

$$\mathcal{D} = \{(f_h, f_u, f_v, f_b, e_h, e_u, e_v, e_b) \in V \times V^* \mid$$
$$\begin{bmatrix} f_h \\ f_u \\ f_v \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} & 0 \\ \mathrm{d} & 0 & -\frac{1}{*h}\mathrm{d}(*v) \\ 0 & \frac{1}{*h}\mathrm{d}(*v) & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_u \\ e_v \end{bmatrix}; \tag{2.69}$$
$$\begin{bmatrix} f_b \\ e_b \\ e_v' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{*h} \end{bmatrix} \begin{bmatrix} e_u \mid_{\partial Z} \\ e_h \mid_{\partial Z} \\ e_v \mid_{\partial Z} \end{bmatrix}.$$

is a Dirac structure, that is $\mathcal{D} = \mathcal{D}^\perp$, where $\perp$ is with respect to (2.68).
In terms of shallow water equations with an additional velocity component the above terms would correspond to

$$\begin{aligned}
f_h &= -\frac{\partial}{\partial t}h(z,t), e_h = \delta_h \mathcal{H} = (\frac{1}{2}((*u)(*u) + (*v)(*v)) + g(*h)) \\
f_u &= -\frac{\partial}{\partial t}u(z,t), e_u = \delta_u \mathcal{H} = (*h)(*u) \\
f_v &= -\frac{\partial}{\partial t}v(z,t), e_v = \delta_v \mathcal{H} = (*h)(*v) \\
f_b &= \delta_u \mathcal{H} \mid_{\partial W}, e_b = -\delta_h \mathcal{H} \mid_{\partial W}, \\
e_v' &= \frac{1}{*h}\delta_v \mathcal{H} \mid_{\partial W}.
\end{aligned} \tag{2.70}$$

with the Hamiltonian given as

$$\mathcal{H} = \int_Z \frac{1}{2}\left((*u)h(*u) + (*u)h(*u)\right) + \frac{1}{2}g(*h)h.$$

Substituting (2.70) into (2.69), we obtain the equations (2.66).

Proof. The proof is based on the skew symmetric term in the 3×3 matrix and also that the boundary variable e_v' in (2.69)does not contribute to the bilinear form (2.68) and also follows a procedure as in [61]. □

Remark 2.19. The Dirac structure above is no more a constant Dirac structure as it depends on the energy variables h, u and v. Moreover, we will also see that of the three boundary variables f_b, e_b and e_v', only f_b and e_b play a role in the power exchange through the boundary as will be seen in the expression for energy balance. We consider e_v' as the third boundary variable instead of $e_v \mid_{\partial W}$ because to study interconnections of such systems we would like to consider v as the boundary variable instead of hv at the boundary as will be shown later in Chapter 3.

Energy Balance

It follows from the power-conserving property of a Dirac structure that the modified Stokes-Dirac structure defined above has the property

$$\int_W (e_h \wedge f_h + e_u \wedge f_u + e_v \wedge f_v) + \int_{\partial W} e_b \wedge f_b = 0,$$

and hence we can get the energy balance

$$\frac{d}{dt}\mathcal{H} = \int_{\partial W} e_b \wedge f_b,$$

which can also be seen by the following

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \int_Z [\delta_h\mathcal{H} \wedge \frac{\partial h}{\partial t} + \delta_u\mathcal{H} \wedge \frac{\partial u}{\partial t} + \delta_v\mathcal{H} \wedge \frac{\partial v}{\partial t}] \\
&= -\int_Z d[\delta_h\mathcal{H}\wedge\delta_u\mathcal{H}] \\
&= \int_{\partial Z} \delta_h\mathcal{H}\wedge\delta_u\mathcal{H} \\
&= \int_{\partial Z} e_b \wedge f_b \\
&= \tilde{h}\tilde{u}(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \mid_0^L \\
&= (\tilde{u}(\frac{1}{2}\tilde{h}\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2)) \mid_0^L + (\tilde{u}(\frac{1}{2}g\tilde{h}^2)) \mid_0^L .
\end{aligned}$$

As in the previous case the first term in last line of the above expression for energy balance corresponds to the energy flux (the total energy times the velocity) through the boundary and the second term is the work done by the hydrostatic pressure given by pressure times the velocity. It is also seen that the boundary variables which contribute to the power at the boundary are f_b and e_b and the third boundary variable e_v' does not contribute to it.

3

Interconnections of port-Hamiltonian Systems

"... for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child." -William Clifford.

In the previous chapter we have defined the notion of a Dirac structure and we have seen how it formalizes a power-conserving interconnection. Using this notion of Dirac structure we defined implicit port-Hamiltonian systems, both finite and infinite-dimensional in nature, which describe energy-conserving systems with external variables. In this chapter we focus on interconnections of such systems. We define what is a power-conserving interconnection, defined on the space of external variables. Since the interconnection preserves the power in the system, we can say that the resulting system would again be energy-conserving. It can also be shown that the resulting system can be described again as a port-Hamiltonian system. This property is useful when modeling energy-conserving systems using a modular approach, where the system is thought of as an interconnection of a number of sub-systems. This has the advantage in the sense that subsystems are smaller and easier to model than the system as a whole. It is also natural from an engineering point of view to regard systems as being composed of different subsystems, from different physical domains. The decomposition can be useful for analyzing the overall behavior of the system. Furthermore, because of the modularity, the modeling process can be performed in an iterative manner, gradually refining the model by adding other subsystems. Also from a control point of view the interconnection approach is important, since implementing a control law or controlling a system is generally done by interconnecting the given system with an external device (a controller) via the external-port variables. In fact, control can be seen in a natural way as the interconnection of the system with other subsystems (the controller) through the port variables. An immediate example is the control by interconnection of

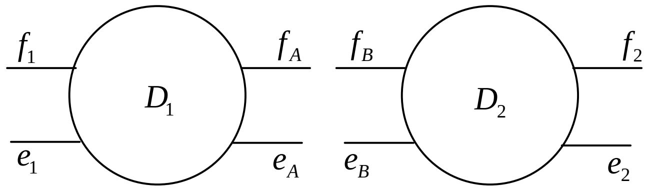


Figure 3.1: The composition of $\mathcal{D}_1$ and $\mathcal{D}_2$

port-Hamiltonian systems, where the plant port-Hamiltonian system is connected to a controller port-Hamiltonian system via a feedback loop, such that the interconnected system has the desired stability properties.

In this chapter we study compositions of Dirac structures and show that the composition is again a Dirac structure and hence the interconnected system again defines a port-Hamiltonian system. The energy of the interconnected system is the total energy of the interconnected system, that is, the sum of energy functions of the individual subsystems. We study composition of finite-dimensional Dirac structures with finite-dimensional Dirac structures and composition of infinite-dimensional systems defined with respect to a Stokes-Dirac structure with Stokes-Dirac structures. Finally, we also study interconnections which are mixed in nature, that is interconnections of finite-dimensional systems with infinite-dimensional systems. Also we study a case of interconnection of two infinite-dimensional systems interconnected to each other through a *distributed* finite-dimensional systems. The concept of interconnection is also extended to systems with dissipation.

3.1 Finite-dimensional systems

In this section we investigate the composition or interconnection properties of finite-dimensional Dirac structures. We recall here the results on composition of two finite-dimensional Dirac structures with partially shared variables.

Thus consider a Dirac structure $\mathcal{D}_1$ on a product space $\mathcal{F}_1 \times \mathcal{F}$ of two linear spaces $\mathcal{F}_1$ and $\mathcal{F}$, and another Dirac structure $\mathcal{D}_2$ on a product space $\mathcal{F} \times \mathcal{F}_2$, with also $\mathcal{F}_2$ being a linear space. The linear space $\mathcal{F}$ is the space of shared flow variables, and $\mathcal{F}^*$ the space of shared effort variables; see Figure 3.1.

In order to compose $\mathcal{D}_1$ and $\mathcal{D}_2$ a problem arises of *sign* convention for the power flow corresponding to the power variables $(f, e) \in \mathcal{F} \times \mathcal{F}^*$. Indeed, if $< e \mid f >$ denotes *incoming* power, then for

$$(f_1, e_1, f_A, e_A) \in \mathcal{D}_1 \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F} \times \mathcal{F}^*,$$

the term $< e_A \mid f_A >$ denotes the incoming power in $\mathcal{D}_1$ due to the power variables $(f_A, e_A) \in \mathcal{F} \times \mathcal{F}^*$, while for

$$(f_B, e_B, f_2, e_2) \in \mathcal{D}_2 \subset \mathcal{F} \times \mathcal{F}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*,$$

the term $< e_B \mid f_B >$ denotes the incoming power in $\mathcal{D}_2$. Clearly, the *incoming* power in $\mathcal{D}_1$ due to the power variables in $\mathcal{F} \times \mathcal{F}^*$ should equal the *outgoing* power from $\mathcal{D}_2$. Thus we cannot simply equate the flows f_A and f_B and the efforts e_A and e_B, but instead we define the interconnection constraints as

$$\begin{aligned} f_A &= -f_B \quad \in \mathcal{F} \\ e_A &= \quad e_B \quad \in \mathcal{F}^*. \end{aligned} \tag{3.1}$$

Therefore the *composition* of the Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_2$, denoted $\mathcal{D}_1 \parallel \mathcal{D}_2$, is defined as

$$\begin{aligned} \mathcal{D}_1 \parallel \mathcal{D}_2 \quad := \quad & \{(f_1, e_1, f, e) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid \exists (f, e) \in \mathcal{F} \times \mathcal{F}^* \text{ s.t.} \\ & (f_1, e_1, f, e) \in \mathcal{D}_1 \text{ and } (-f, e, f_2, e_2) \in \mathcal{D}_2\}\,. \end{aligned} \tag{3.2}$$

The fact that the composition of two finite-dimensional Dirac structures is again a Dirac structure has been proved before in [7, 59]. We just recall here the proof of the same here.

Theorem 3.1. *[7] Let $\mathcal{D}_1, \mathcal{D}_2$ be Dirac structures as in Definition 2.1 (defined with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F} \times \mathcal{F}^*$, respectively $\mathcal{F} \times \mathcal{F}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, and their bilinear forms). Then $\mathcal{D}_1 \parallel \mathcal{D}_2$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$.*

Proof. Consider $\mathcal{D}_1, \mathcal{D}_2$ defined in matrix kernel representation by

$$\begin{aligned} \mathcal{D}_1 &= \{(f_1, e_1, f, e) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F} \times \mathcal{F}^* \mid F_1 f_1 + E_1 e_1 + F f + E e = 0\} \\ \mathcal{D}_2 &= \{(f', e', f_2, e_2) \in \mathcal{F} \times \mathcal{F}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid F' f' + E' e' + F_2 f_2 + E_2 e_2 = 0\}. \end{aligned}$$

In the following we shall make use of the following basic fact from linear algebra:

$$[(\exists \lambda \text{ s.t. } A\lambda = b)] \Leftrightarrow [\forall \alpha \text{ s.t. } \alpha^T A = 0 \Rightarrow \alpha^T b = 0]. \tag{3.3}$$

Note that $\mathcal{D}_1, \mathcal{D}_2$ are alternatively given in matrix image representation as

$$\mathcal{D}_1 = \text{im} \begin{bmatrix} E_1^T \\ F_1^T \\ E^T \\ F^T \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{D}_2 = \text{im} \begin{bmatrix} 0 \\ 0 \\ E'^T \\ F'^T \\ E_2^T \\ F_2^T \end{bmatrix}. \tag{3.4}$$

Hence,

$$(f_1,e_1,f_2,e_2) \in \mathcal{D}_1 \parallel \mathcal{D}_2 \Leftrightarrow \exists \lambda, \lambda' \text{ s.t.} \begin{bmatrix} f_1 \\ e_1 \\ 0 \\ 0 \\ f_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} E_1^T & 0 \\ F_1^T & 0 \\ E^T & E'^T \\ F^T & -F'^T \\ 0 & F_2^T \\ 0 & E_2^T \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda' \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \forall (\beta_1,\alpha_1,\beta',\alpha',\beta_2,\alpha_2) \text{ s.t. } (\beta_1^T \alpha_1^T \beta_2' \alpha_2' \beta_2^T \alpha_2^T) \begin{bmatrix} E_1^T & 0 \\ F_1^T & 0 \\ E^T & E'^T \\ F^T & -F'^T \\ 0 & F_2^T \\ 0 & E_2^T \end{bmatrix} = 0,$$

$$\beta_1^T f_1 + \alpha_1^T e_1 + \beta_2^T f_2 + \alpha_2^T e_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow \forall (\alpha_1,\beta_1,\alpha',\beta',\alpha_2,\beta_2) \text{ s.t. } \begin{bmatrix} F_1 & E_1 & F^T & E^T & 0 & 0 \\ 0 & 0 & -F'^T & E'^T & F_2 & E_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha' \\ \beta' \\ \alpha_2 \\ \beta_2 \end{bmatrix} = 0,$$

$$\beta_1^T f_1 + \alpha_1^T e_1 + \beta_2^T f_2 + \alpha_2^T e_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow \forall (\alpha_1,\beta_1,\alpha_2,\beta_2) \in \mathcal{D}_1 \parallel \mathcal{D}_2, \quad \beta_1^T f_1 + \alpha_1^T e_1 + \beta_2^T f_2 + \alpha_2^T e_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow (f_1,e_1,f_2,e_2) \in (\mathcal{D}_1 \parallel \mathcal{D}_2)^\perp.$$

Thus $\mathcal{D}_1 \parallel \mathcal{D}_2 = (\mathcal{D}_1 \parallel \mathcal{D}_2)^\perp$, and so it is a Dirac structure.

Remark 3.2. Instead of the canonical interconnection constraints $f_A = -f_B$, $e_A = e_B$ (cf. Equation (3.1)) another standard power-conserving interconnection is the 'gyrative' interconnection

$$f_A = e_B, \quad f_B = -e_A. \tag{3.5}$$

(The standard feedback interconnection, regarding f_A, f_B as *inputs*, and e_A, e_B as *outputs*, is of this type.) Composition of two Dirac structures $\mathcal{D}_1, \mathcal{D}_2$ by this gyrative interconnection also results in a Dirac structure. In fact, the gyrative interconnection of $\mathcal{D}_1$ and $\mathcal{D}_2$ equals the interconnection $\mathcal{D}_1 \parallel \mathcal{I} \parallel \mathcal{D}_2$, where $\mathcal{I}$ is the gyrative (or *symplectic*) Dirac structure

$$f_{IA} = -e_{IB}, \quad f_{IB} = -e_{IA}, \tag{3.6}$$

interconnected to $\mathcal{D}_1$ and $\mathcal{D}_2$ via the canonical interconnections
$f_{IA} = -f_A, e_{IA} = e_A$ and $f_{IB} = -f_B, e_{IB} = e_B$.

Since we now know that the composition of two Dirac structures is a Dirac structure, it readily follows that the power-conserving interconnection of a number of Dirac structures is again a Dirac structure. Let us consider l Dirac structures $\mathcal{D}_k \subset \mathcal{F}_k \times \mathcal{F}_k^* \times \mathcal{F}_{IK} \times \mathcal{F}_{IK}^*$, $k = 1, ..., l$, interconnected to each other via an interconnection Dirac structure $\mathcal{D}_I \subset \mathcal{F}_{I1} \times \mathcal{F}_{I1}^* \times ... \times \mathcal{F}_{I1} \times \mathcal{F}_{I1}^*$. This can be regarded as the composition of the product Dirac structure $\mathcal{D}_1 \times ... \times \mathcal{D}_l$ with the interconnection Dirac structure $\mathcal{D}_I$. Hence by the above theorem the overall interconnection is again a Dirac structure.

It is immediate that the composition of Dirac structures is *associative* in the following sense. Given two Dirac structures $\mathcal{D}_1 \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{D}_2 \subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$ their composition $\mathcal{D}_1 \parallel \mathcal{D}_2$. Now compose the composed Dirac structure $\mathcal{D}_1 \parallel \mathcal{D}_2$ with another Dirac structure $\mathcal{D}_3 \subset \mathcal{F}_3 \times \mathcal{F}_3^* \times \mathcal{F}_4 \times \mathcal{F}_4^*$, resulting in the composition $(\mathcal{D}_1 \parallel \mathcal{D}_2) \parallel \mathcal{D}_3$. It can easily be checked that the same composed Dirac structure results by first composing $\mathcal{D}_2$ with $\mathcal{D}_3$, and then composing the outcome with $\mathcal{D}_1$, that is

$$(\mathcal{D}_1 \parallel \mathcal{D}_2) \parallel \mathcal{D}_3 = \mathcal{D}_1 \parallel (\mathcal{D}_2 \parallel \mathcal{D}_3) = \mathcal{D}_1 \parallel \mathcal{D}_2 \parallel \mathcal{D}_3.$$

3.1.1 Composition of Dirac structure and a resistive relation

Proposition 3.3. *Let $\mathcal{D}$ be a Dirac structure defined with respect to $\mathcal{F}_s \times \mathcal{F}_s^* \times \mathcal{F}_R \times \mathcal{F}_R^*$. Furthermore, let $\mathcal{R}$ be a resistive relation defined with respect to $\mathcal{F}_R \times \mathcal{F}_R^*$ given by*

$$R_f f_R + R_e e_R = 0,$$

where the square matrices R_f and R_e satisfy the symmetry and semi-positive definiteness condition

$$R_f R_e^T = R_e R_f^T \geq 0.$$

Define the composition $\mathcal{D} \parallel \mathcal{R}$ of the Dirac structure and the resistive relation in the same way as the composition of two Dirac structures. Then

$$(\mathcal{D} \parallel \mathcal{R})^{\perp} = \mathcal{D} \parallel -\mathcal{R},$$

where $-\mathcal{R}$ denotes the pseudo-resistive relation given by

$$R_f f_R - R_e e_R = 0.$$

($-\mathcal{R}$ is called a pseudo-resistive relation since it corresponds to negative instead of positive resistance).

Proof. We follow the same steps as in the proof that the composition of two Dirac structures is again a Dirac structure, Theorem 3.1 (where we take $\mathcal{F}_1 = \mathcal{F}_S$, $\mathcal{F}_2 = \mathcal{F}_R$, and $\mathcal{F}_3$ void). Because of the sign difference in the definition of a resistive relation as compared with the definition of a Dirac structure we immediately obtain the stated proposition. $\square$

3.1.2 Port-Hamiltonian system with dissipation

Using the result in Proposition 3.3, we can extend the kernel representation (2.36) for a port-Hamiltonian system to a system with dissipation. The Dirac structure with dissipation would now be given as

$$\begin{aligned}\mathcal{D} = \; & \{(f_s, e_s, f_R, e_R, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F}_R \times \mathcal{F}_R^* \times \mathcal{F} \times \mathcal{F}^* \mid \\ & F_S f_s + E_S e_s + F_R f_R + E_R e_R + Ff + Ee = 0\},\end{aligned}$$

with

$$(i) \quad E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + EF^T + FE^T = 0$$

$$(ii) \quad \operatorname{rank} F_S \vdots E_s \vdots F_R \vdots E_R \vdots F \vdots E = \dim(\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}),$$

with the flow and effort variables connected to the resistive elements related as

$$f_R = -Re_{R.}$$

Then the port-Hamiltonian system with dissipation is given by the set of equations

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) - F_R R e_R(t) + E_R e_R(t) + Ff(t) + Ee(t) = 0$$

and it satisfies the energy balance

$$\begin{aligned}\frac{dH}{dt}(x(t)) &= < \frac{\partial H}{\partial x}(x(t) \mid \dot{x}(t) > \\ &= e_c^T f_c + e_I^T f_I - e_R R e_R \\ &\leq e_c^T f_c + e_I^T f_I.\end{aligned}$$

In the same way as the composition of two Dirac structures we can also treat the problem of composition of two resistive relations.

Proposition 3.4. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be resistive relations as in Proposition 3.3, defined with respect to $\mathcal{F}_{R1} \times \mathcal{F}_{R1}^* \times \mathcal{F} \times \mathcal{F}^*$, respectively $\mathcal{F}_{R2} \times \mathcal{F}_{R2}^* \times \mathcal{F} \times \mathcal{F}^*$and their bilinear forms, with $\mathcal{F} \times \mathcal{F}^*$ being the space of shared flow and effort variables between $\mathcal{R}_1$ and $\mathcal{R}_2$. Then $\mathcal{R}_1 \parallel \mathcal{R}_2$ is again a resistive relation with the property that*

$$(\mathcal{R}_1 \parallel \mathcal{R}_2)^\perp = -\mathcal{R}_1 \parallel -\mathcal{R}_2,$$

where again $-\mathcal{R}$ denotes the pseudo resistive relation as stated in the above proposition.

Proof. The proof follows the same steps as in the proof of composition of two Dirac structures, Theorem 3.1 (where we take $\mathcal{F}_1 = \mathcal{F}_{R1}$, $\mathcal{F}_2 = \mathcal{F}$ and $\mathcal{F}_3 = \mathcal{F}_{R2}$). Again because of the sign difference in the definition of a Dirac structure we immediately obtain the stated proposition. □

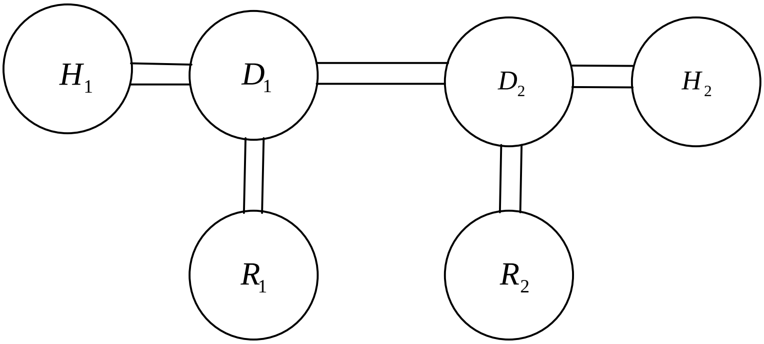


Figure 3.2: $(\mathcal{D}_1 \parallel \mathcal{R}_1) \parallel (\mathcal{D}_2 \parallel \mathcal{R}_2)$

Now suppose we have a port-Hamiltonian system with dissipation (which can be viewed as the composition of a Dirac structure and a resistive relation) and we wish to interconnect it to another port Hamiltonian system with dissipation, what would the total interconnection structure look like? The answer to this question is the following corollary which comes as a result of the above three propositions.

Corollary 3.5. *Let $\mathcal{D}_1 \parallel \mathcal{R}_1$ and $\mathcal{D}_2 \parallel \mathcal{R}_2$ be two Dirac structures interconnected to a resistive relation (each representing a port-Hamiltonian system with dissipation), then the composed structure $(\mathcal{D}_1 \parallel \mathcal{R}_1) \parallel (\mathcal{D}_2 \parallel \mathcal{R}_2)$ (see Figure 3.2) will have a structure of the form $\mathcal{D} \parallel \mathcal{R}$ with the property that $(\mathcal{D} \parallel \mathcal{R})^{\perp} = (\mathcal{D} \parallel -\mathcal{R})$, with $-\mathcal{R}$ again denoting the pseudo resistive relation corresponding to negative resistance. $\mathcal{D}$ is the result of composition of Dirac structures of both systems $\mathcal{D}_1 \parallel \mathcal{D}_2$ and $\mathcal{R}$ is the composition of resistive relations of both the systems $\mathcal{R}_1 \parallel \mathcal{R}_2$. In the Figure H_1 and H_2 are the Hamiltonians of the respective port-Hamiltonian systems with dissipation.*

3.2 Infinite-dimensional systems

Similar to finite-dimensional case one can also investigate interconnections in the infinite-dimensional case. However there is clear distinction in the case of infinite-dimensional systems in the sense that interconnections in infinite-dimensional systems can occur in two ways, either through the spatial domain or through the boundary of the spatial domain of the system or both. The same case holds for composition of infinite-dimensional system given by a Stokes-Dirac structure with a resistive relation where the resistive relation can again either enter through the spatial domain (a part or whole of the spatial domain) of the Stokes-Dirac structure or by terminating some or

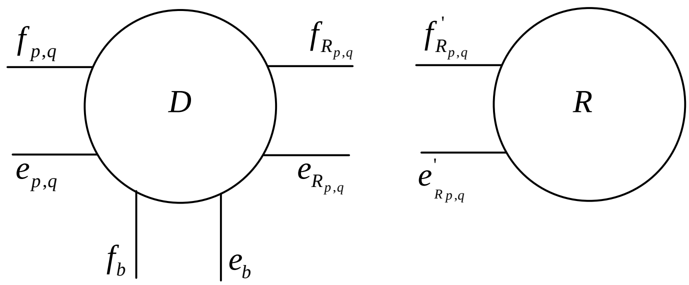


Figure 3.3: The interconnection of $\mathcal{D}$ and $\mathcal{R}$

all of its boundary ports by a resistive relation. The case of terminating the boundary of the system with a 1-D spatial domain by a resistive relation can be considered as a special case of interconnection of mixed finite and infinite-dimensional port-Hamiltonian systems as will be discussed in Section 3.2.1.

3.2.1 Composition of Dirac structure and a resistive relation

We discuss here the composition of a Stokes-Dirac structure and a resistive relation, where the dissipation enters into the system through the spatial domain (part or whole of it). This composition enables us to define an infinite-dimensional port-Hamiltonian system with dissipation. Consider a Stokes-Dirac structure $\mathcal{D}$ defined on a product space $\mathcal{F}_{p,q} \times \mathcal{F}_{R_{p,q}} \times \mathcal{F}_b$[1] and a resistive relation $\mathcal{R}$ on the space $\mathcal{F}_{R_{p,q}}$. The space $\mathcal{F}_{R_{p,q}} \times \mathcal{E}_{R_{p,q}}$ is the space of shared flow and effort variables between $\mathcal{D}$ and $\mathcal{R}$. We then define the standard interconnection constraints as

$$\begin{aligned} f_{Rp} &= -f'_{Rp} \in \mathcal{F}_{Rp}, \quad f_{Rq} = -f'_{Rq} \in \mathcal{F}_{Rq} \\ e_{Rp} &= -e'_{Rp} \in \mathcal{E}_{Rp}, \quad e_{Rq} = -e'_{Rq} \in \mathcal{E}_{Rq}. \end{aligned} \tag{3.7}$$

The composition $\mathcal{D} \parallel \mathcal{R}$ is then defined as (also see Figure 3.3)

$$\begin{aligned} \mathcal{D} \parallel \mathcal{R} := \quad & \{(f_p, f_q, e_p, e_q, f_b, e_b) \in \mathcal{F}_p \times \mathcal{F}_q \times \mathcal{E}_p \times \mathcal{E}_q \times \mathcal{F}_b \times \mathcal{E}_b \mid \\ & \exists (f_{Rp}, f_{Rq}, e_{Rp}, e_{Rq}) \in \mathcal{F}_{Rp} \times \mathcal{F}_{Rq} \times \mathcal{E}_{Rp} \times \mathcal{E}_{Rq} \text{ s.t.} \\ & (f_p, f_q, e_p, e_q, f_{Rp}, f_{Rq}, e_{Rp}, e_{Rq}, f_b, e_b) \in \mathcal{D}, \\ & (-f_{Rp}, -f_{Rq}, e_{Rp}, e_{Rq}) \in \mathcal{R}. \end{aligned}$$

[1]For brevity, we sometimes use the notation $\mathcal{F}_{p,q}$ for $\mathcal{F}_p \times \mathcal{F}_q$, similarly for $\mathcal{E}_{p,q}, \mathcal{F}_{R_{p,q}}, \mathcal{E}_{R_{p,q}}$

Proposition 3.6. *Let $\mathcal{D}$ be a Stokes-Dirac structure defined with respect to $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_{R_{p,q}} \times \mathcal{E}_{R_{p,q}} \times \mathcal{F}_b \times \mathcal{E}_b$ as follows:*

$$\begin{aligned}
\begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} - g_R \begin{bmatrix} f_{Rp} \\ f_{Rq} \end{bmatrix} \\
\begin{bmatrix} e_{Rp} \\ e_{Rq} \end{bmatrix} &= g_R^T \begin{bmatrix} e_p \\ e_q \end{bmatrix} \\
\begin{bmatrix} f_b \\ f_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p \mid_{\partial Z} \\ e_q \mid_{\partial Z} \end{bmatrix}.
\end{aligned} \tag{3.8}$$

Furthermore let $\mathcal{R}$ be a resistive relation defined with respect to $\mathcal{F}_{R_{p,q}} \times \mathcal{E}_{R_{p,q}}$. Let $S* : \Omega^{n-k}(Z) \to \Omega^k(Z)$ be a map satisfying

$$\int_Z e_R \wedge (S * e_R) = \int_Z (S * e_R) \wedge e_R \geq 0, \forall e_R \in \Omega^{n-k}(Z), R \in \mathbb{R}. \tag{3.9}$$

We consider here a typical case where the flows and the efforts of the energy dissipating elements are related as

$$f_R = -S * e_R.$$

Here f_R and e_R correspond to the flows and effort variables of the resistive elements in both the p and q energy domains, i.e.

$$\begin{aligned}
f_R &= \begin{bmatrix} f_{R_p} & f_{R_q} \end{bmatrix}^T \\
e_R &= \begin{bmatrix} e_{R_p} & e_{R_q} \end{bmatrix}^T,
\end{aligned}$$

similarly S also incorporates the dissipation in both the energy domains. Typically S is a block of the form

$$S = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}. \tag{3.10}$$

Defining interconnections of $\mathcal{D}$ and $\mathcal{R}$ in the standard way, we have the composed structure as follows

$$\begin{aligned}
\begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} - g_R \begin{bmatrix} f_{Rp} \\ f_{Rq} \end{bmatrix} \\
\begin{bmatrix} f_{Rp} \\ f_{Rq} \end{bmatrix} &= - \begin{bmatrix} G* & 0 \\ 0 & R* \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} \\
\begin{bmatrix} e_{Rp} \\ e_{Rq} \end{bmatrix} &= g_R^T \begin{bmatrix} e_p \\ e_q \end{bmatrix}.
\end{aligned} \tag{3.11}$$

together with the boundary conditions. The composed structure then has the following property

$$(\mathcal{D} \parallel \mathcal{R})^{\perp} = \mathcal{D} \parallel -\mathcal{R}$$

where $\mathcal{R}$ again is a pseudo resistive relation (corresponding to the negative resistance).

Proof. For simplicity of the proof we assume zero boundary conditions (meaning that all the boundary variables are set to zero). Then the bilinear form on $\mathcal{D} \parallel \mathcal{R}$ is given by

$$<< (f_p^1, f_q^1, e_p^1, e_q^1), (f_p^2, e_p^2, f_q^2, e_q^2) >>:= \int_z [e_p^2 \wedge f_p^1 + e_p^1 \wedge f_p^2 + e_q^2 \wedge f_q^1 + e_q^1 \wedge f_q^2]. \tag{3.12}$$

Part1: Take a $(f_p^1, f_q^1, e_p^1, e_q^1) \in \mathcal{D} \parallel \mathcal{R}$ and take any other $(f_p^2, f_q^2, e_p^2, e_q^2) \in \mathcal{D} \parallel (-\mathcal{R})$. By substituting (3.8) into (3.12), the right hand side of (3.12) becomes

$$\int_Z [e_p^2 \wedge (\mathrm{d}e_q^1 + G * e_p^1) + e_p^1 \wedge (\mathrm{d}e_q^2 - G * e_p^2) + e_q^2 \wedge (\mathrm{d}e_p^1 + R * e_q^1) + e_q^1 \wedge (\mathrm{d}e_p^2 - R * e_q^2)]$$

$$= \int_Z [e_p^2 \wedge \mathrm{d}e_q^1 + e_p^2 \wedge G * e_p^1 + e_p^1 \wedge \mathrm{d}e_q^2 - e_p^1 \wedge G * e_p^2 + e_q^2 \wedge \mathrm{d}e_p^1 + e_q^2 \wedge R * e_q^1 + e_q^1 \wedge \mathrm{d}e_p^2 - e_q^1 \wedge R * e_q^2]. \tag{3.13}$$

We now use the following properties of the exterior derivative and the Hodge star operator

$$\begin{aligned} &\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta \\ &\alpha \wedge \beta = \beta \wedge \alpha \\ &\alpha \wedge (\beta * \gamma) = (\beta * \alpha) \wedge \gamma. \end{aligned}$$

then, using the above properties and the Stokes' theorem, Equation (3.13) can be written as

$$\begin{aligned} &\textstyle\int_Z [\mathrm{d}(e_p^2 \wedge e_q^1) - \mathrm{d}e_p^2 \wedge e_q^1 + G * e_p^2 \wedge e_p^1 + \mathrm{d}(e_p^1 \wedge e_q^2) - \mathrm{d}e_p^1 \wedge e_q^2 \\ &-G * e_p^2 \wedge e_p^1 + \mathrm{d}e_p^1 \wedge e_q^2 + R * e_q^2 \wedge e_q^1 + \mathrm{d}e_p^2 \wedge e_q^1 - R * e_q^2 \wedge e_q^1] = 0 \end{aligned}$$

Hence $(\mathcal{D} \parallel -\mathcal{R}) \subset (\mathcal{D} \parallel \mathcal{R})^\perp$

PartII: Let $(f_p^1, f_q^1, e_p^1, e_q^1) \in \mathcal{D} \parallel \mathcal{R}$ and let $(f_p^2, f_q^2, e_p^2, e_q^2) \in (\mathcal{D} \parallel \mathcal{R})^\perp$, hence the right hand side of (3.12) is zero for these elements and hence by substitution, we have

$$\begin{aligned} &\textstyle\int_Z [e_p^2 \wedge (de_q^1 + G * e_p^1) + e_p^1 \wedge f_p^2 + e_q^2 \wedge (de_p^1 + R * e_q^1) + e_q^1 \wedge f_q^2] = 0 \\ \Rightarrow \quad &\textstyle\int_Z [e_p^2 \wedge de_q^1 + e_p^2 \wedge G * e_p^1 + e_p^1 \wedge f_p^2 + e_q^2 \wedge de_p^1 + e_q^2 \wedge R * e_q^1 + e_q^1 \wedge f_q^2] = 0. \end{aligned}$$

Now, again using the above mentioned properties of the exterior derivative and the Hodge star operator,we get

$$\begin{aligned} &\int_Z [d(e_p^2 \wedge e_q^1) - \mathrm{d}e_p^2 \wedge e_q^1 + G * e_p^2 \wedge e_p^1 + e_p^1 \wedge f_p^2 + \\ &\mathrm{d}(e_q^2 \wedge e_p^1) - \mathrm{d}e_q^2 \wedge e_p^1 + R * e_q^2 \wedge e_q^1 + e_q^1 \wedge f_q^2 = 0]. \end{aligned}$$

Since, we assume zero boundary conditions and applying the Stokes' theorem the above expression can be written as

$$\int_z -(\mathrm{d}e_p^2 - R * e_q^2) \wedge e_q^1 - (\mathrm{d}e_q^2 - G * e_p^2) \wedge e_p^1 + f_p^2 \wedge e_p^1 + f_q^2 \wedge e_q^1 = 0,$$

which implies that

$$\begin{aligned} f_p^2 &= \mathrm{d}e_q^2 - G * e_p^2 \\ f_q^2 &= \mathrm{d}e_p^2 - R * e_q^2, \end{aligned}$$

showing that $(f_p^2, f_q^2, e_p^2, e_q^2) \in \mathcal{D} \parallel (-\mathcal{R})$, which means that $(\mathcal{D} \parallel \mathcal{R})^\perp \subset \mathcal{D} \parallel (-\mathcal{R})$, completing the proof □

Remark 3.7. Equation (3.11), defines an infinite-dimensional port-Hamiltonian system with dissipation. The port-Hamiltonian system with dissipation now satisfies the energy balance inequality

$$\begin{aligned} \frac{dH}{dt} &= \int_{\partial Z} f_b \wedge e_b - \int_Z e_R \wedge S(e_R) \\ &\leq \int_{\partial Z} f_b \wedge e_b. \end{aligned}$$

Example 3.8 (A transmission line with dissipation). Consider the dynamics of the transmission line as in (2.55) but now with dissipation in the line. The dynamics are then given by

$$\begin{aligned} \frac{\partial q}{\partial t} &= -\mathrm{d}\left(\frac{*\phi(t,z)}{L(z)}\right) - G(z)\left(\frac{q(t,z)}{C(z)}\right) \\ \frac{\partial \phi}{\partial t} &= -\mathrm{d}\left(\frac{*q(t,z)}{C(z)}\right) - R(z)\left(\frac{\phi(t,z)}{L(z)}\right). \end{aligned} \tag{3.14}$$

with $G(z)$, $R(z)$ respectively the distributed conductance and resistance of the line. Now, in terms of the definition of the resistive relation in Proposition 3.6, the matrix S in (3.10) would accommodate the distributed conductance $G(z)$ and the distributed resistance $R(z)$ of the line. The corresponding Dirac structure with dissipation is given by

$$\begin{bmatrix} -\frac{\partial q}{\partial t} \\ -\frac{\partial \phi}{\partial t} \end{bmatrix} = \left(\begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} - \begin{bmatrix} G* & 0 \\ 0 & R* \end{bmatrix} \right) \begin{bmatrix} \frac{*q(t,z)}{C(z)} \\ \frac{*\phi(t,z)}{L(z)} \end{bmatrix}, \tag{3.15}$$

together with boundary voltages and currents. The transmission line with dissipation then satisfies the following energy balance inequality

$$\frac{dH}{dt} = -\int_Z \left\{ GV^2(z,t) + RI^2(z,t) \right\} + V(L,t)I(L,t) - V(0,t)I(0,t),$$

where
$$V(t,z) = \tfrac{*q(t,z)}{C(z)}, \quad I(t,z) = \tfrac{*\phi(t,z)}{L(z)} \;\; ,$$
are the distributed voltages and currents respectively.

3.2.2 Composition of Dirac structures

In a similar way we can also view the interconnection of two Stokes-Dirac structures through the spatial domain, in which case we replace the resistive relation in the above proposition with a Stokes-Dirac structure. We consider a Stokes-Dirac structure $\mathcal{D}_1$ on a product space $\mathcal{F}_{p,q} \times \mathcal{F}_{b1}$ and another Dirac structure $\mathcal{D}_2$ on the product space $\mathcal{F}_{p,q} \times \mathcal{F}_{b2}$. The space $\mathcal{F}_{p,q}$ is the space of shared flow variables and $\mathcal{E}_{p,q}$ the space of shared effort variables. The composition of these two Stokes-Dirac structures is then to define the interconnection constraints as

$$f_p = -f_p' \in \mathcal{F}_p, \quad f_q = -f_q' \in \mathcal{F}_q$$

$$e_p = e_p' \in \mathcal{E}_p, \qquad e_q = e_q' \in \mathcal{E}_q.$$

The composition $\mathcal{D}_1 \parallel \mathcal{D}_2$ of the two Stokes-Dirac structures is defined as

$$\begin{aligned}
\mathcal{D}_1 \parallel \mathcal{D}_2 := \quad & \{(f_{b1}, e_{b1}, f_{b2}, e_{b2}) \in \mathcal{F}_{b1} \times \mathcal{E}_{b1} \times \mathcal{F}_{b2} \times \mathcal{E}_{b2} \mid \\
& \exists (f_p, f_q, e_p, e_q) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \text{ s.t.} \\
& (f_p, f_q, e_p, e_q, f_{b_1}, e_{b_1}) \in \mathcal{D}_1 \text{ and } (f_p, f_q, e_p, e_q, f_{b_2}, e_{b_2}) \in \mathcal{D}_2.
\end{aligned}$$

Proposition 3.9. *Let $\mathcal{D}_{1,}$ $\mathcal{D}_2$ be a stokes-Dirac structures defined with respect to $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_{1b} \times \mathcal{E}_{1b}$, respectively $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_{2b} \times \mathcal{E}_{2b}$ and their respective bilinear forms. Then $\mathcal{D}_1 \parallel \mathcal{D}_2$ is a Dirac structure defined with respect to the bilinear form on $\mathcal{F}_{1b} \times \mathcal{E}_{1b} \times \mathcal{F}_{2b} \times \mathcal{E}_{2b}$, the boundary variables.*

Proof. The proof follows the same procedure as in Proposition 3.6 where now the bilinear form is defined on the space of boundary variables as

$$\begin{aligned}
& << (f_{b1}^1, e_{b1}^1, f_{b2}^1, e_{b2}^1), (f_{b1}^2, e_{b1}^2, f_{b2}^2, e_{b2}^2) >>:= \\
& \int_{\partial z} [e_{b1}^2 \wedge f_{b1}^1 + e_{b1}^1 \wedge f_{b1}^2 + e_{b2}^2 \wedge f_{b2}^1 + e_{b2}^1 \wedge f_{b2}^2].
\end{aligned}$$

The rest of the proof follows the same procedure, by proving the following two inclusions

$$(\mathcal{D}_1 \parallel \mathcal{D}_2) \subset (\mathcal{D}_1 \parallel \mathcal{D}_2)^{\perp}$$
$$(\mathcal{D}_1 \parallel \mathcal{D}_2)^{\perp} \subset (\mathcal{D}_1 \parallel \mathcal{D}_2).$$

□

3.2.3 Interconnections through the boundary

So far we have looked at interconnections or composition of Stokes-Dirac structures through the spatial domain. We now focus on interconnections of two Stokes-Dirac structures through the boundary. Consider a Stokes-Dirac structure $\mathcal{D}_1$ on a product space $\mathcal{F}_{p,q} \times \mathcal{F}_b$ and another Dirac structure $\mathcal{D}_2$ on the product space $\mathcal{F}'_{p,q} \times \mathcal{F}_b$. Since the interconnection takes place through the boundary of the system, the space of shared flow and effort variables is $\mathcal{F}_b$, $\mathcal{E}_b$ respectively. We define the interconnection constraints as follows

$$\begin{aligned} f_b &= -f'_b \in \mathcal{F}_b \\ e_b &= e'_b \in \mathcal{E}_b. \end{aligned} \tag{3.16}$$

The composition $\mathcal{D}_1 \parallel \mathcal{D}_2$ is then defined as

$$\begin{aligned} \mathcal{D}_1 \parallel \mathcal{D}_2 := \quad & \{(f_{p_1}, f_{q_1}, e_{p_1}, e_{q_1}, f_{p_2}, f_{q_2}, e_{p_2}, e_{q_2}) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q} \mid \\ & \exists (f_b, e_b) \in \mathcal{F}_b \times \mathcal{E}_b \text{ s.t.} \\ & (f_p, f_q, e_p, e_q, f_b, e_b) \in \mathcal{D}_1 \text{ and } (f'_p, f'_q, e'_p, e'_q, -f_b, e_b) \in \mathcal{D}_2. \end{aligned}$$

This yields the following bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q}$:

$$\begin{aligned} &<< (f^a_{p_1}, f^a_{q_1}, e^a_{p_1}, e^a_{q_1}, f^a_{p_2}, f^a_{q_2}, e^a_{p_2}, e^a_{q_2}), (f^b_{p_1}, f^b_{q_1}, e^b_{p_1}, e^b_{q_1}, f^b_{p_2}, f^b_{q_2}, e^b_{p_2}, e^b_{q_2}) >> \\ &:= \int_{Z_1} [e^b_{p_1} \wedge f^a_{p_1} + e^a_{p_1} \wedge f^b_{p_1} + e^b_{q_1} \wedge f^a_{q_1} + e^a_{q_1} \wedge f^b_{q_1}] \\ &+ \int_{Z_2} [e^b_{p_2} \wedge f^a_{p_2} + e^a_{p_2} \wedge f^b_{p_2} + e^b_{q_2} \wedge f^a_{q_2} + e^a_{q_2} \wedge f^b_{q_2}]. \end{aligned} \tag{3.17}$$

Proposition 3.10. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be Stokes-Dirac structures as above defined respectively with respect to $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b$ and $\mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b$. Then the composition $\mathcal{D} = \mathcal{D}_1 \parallel \mathcal{D}_2$ is a Dirac structure defined with respect to the bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q}$ given by (3.17)*

Proof. The proof again follows the same procedure as in the above proposition and hence we omit it here. □

Remark 3.11. One must observe here that the interconnection constraints (3.16) involve whole of the boundary. On the other hand there can be cases where we the interconnection takes place through a part of the boundary. In cases such as this the interconnection constraints and the bilinear form would vary accordingly.

Example 3.12 (Interconnections of cross canals). This is a typical example of interconnections of infinite-dimensional systems through a part of the boundary. Consider interconnection of three canals as shown in the Figure

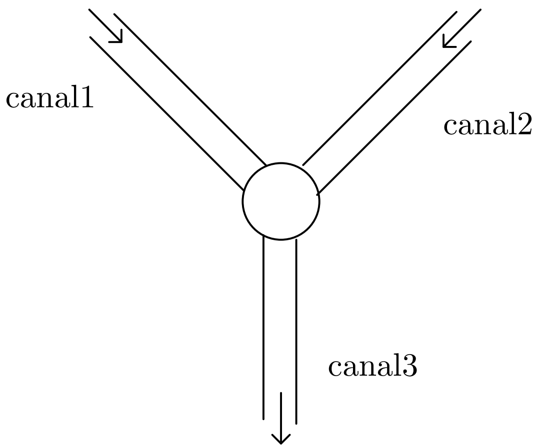


Figure 3.4: Interconnections of cross canals

3.4. The dynamics of each of the canals are given by the shallow water equations given by (2.64)

$$\begin{bmatrix} -\frac{\partial}{\partial t} h_i(z,t) \\ -\frac{\partial}{\partial t} u_i(z,t) \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(*u_i)(*u_i) + g * h_i \\ *h_i * u_i \end{bmatrix}; \quad i = 1,2,3$$

together with the boundary variables

$$\begin{aligned} f_{bi} &= *h_i * u_i \mid_b = \tilde{h}\tilde{u} \mid_b \\ e_{bi} &= \tfrac{1}{2}(*u_i)(*u_i) + g * h_i \mid_b = \tfrac{1}{2}\tilde{u}^2 + g\tilde{h} \mid_b . \end{aligned}$$

Denote by $f_{0i}, e_{0i}, i = 1,2,3$ the boundary flows and efforts at the intersection of the three canals: The interconnection constraints are

$$\begin{aligned} & e_{01} = e_{02} = e_{03} \\ & f_{01} + f_{02} + (-f_{03}) = 0, \end{aligned} \tag{3.18}$$

or written out

$$\begin{aligned} & \tfrac{1}{2}(\tilde{u}_{01}^2 + g\tilde{h}_{01}) = \tfrac{1}{2}(\tilde{u}_{01}^2 + g\tilde{h}_{01}) = \tfrac{1}{2}(\tilde{u}_{01}^2 + g\tilde{h}_{01}) \\ & \tilde{h}_{01}\tilde{u}_{01} + \tilde{h}_{02}\tilde{u}_{02} + (-\tilde{h}_{03}\tilde{u}_{03}) = 0. \end{aligned} \tag{3.19}$$

It can easily be seen that the interconnection constraints (3.18) are indeed power-conserving and thus the interconnected system again defines a port-Hamiltonian system. Physically these constraints mean that the mass is conserved at the intersection of the canals and that the Bernoulli function remains the same.

Remark 3.13. In case we have an additional velocity component in the canals as in (2.66), then at the intersection the constraints on the boundary term arising due to the additional velocity component would be

$$\tilde{v}_{01} = \tilde{v}_{02} = \tilde{v}_{03}.$$

This justifies the reason for considering $*v = \tilde{v}$ as the boundary variables instead of hv as stated in Equation (2.70) of Proposition 2.18.

Remark 3.14. If we consider infinite-dimensional Dirac structures defined on Hilbert spaces, then the compositional property is not immediate, as shown in [22, 17]. Necessary and sufficient conditions have been derived in [22] for the composition of two or more Dirac structures on Hilbert spaces to again define a Dirac structure. The infinite-dimensional Dirac structures we focus on here are of a particular kind, which we call the Stokes-Dirac structure, which are defined on spaces of differential forms. We have shown that a power-conserving interconnection of a number of Stokes-Dirac structures is again a Stokes-Dirac structure. Now, to relate this to the Hilbert space setting, our conjecture would be that composition of Stokes-Dirac structure would satisfy the necessary and sufficient conditions as derived in [22] for the composition to again define a Stokes-Dirac structure.

3.3 Mixed port-Hamiltonian systems

Mixed port-Hamiltonian systems arise by interconnections of finite-dimensional port-Hamiltonian systems with infinite-dimensional port-Hamiltonian systems (see also [24, 51]). We here study interconnections of such systems and show that the composition of their Dirac structures is again a Dirac structure, hence the interconnected system is again a port-Hamiltonian system. Typical example of such an interconnection is a power-drive consisting of a power converter, transmission line and electrical machine.

3.3.1 Interconnection of mixed finite and infinite-dimensional systems

We consider the composition of two Dirac structures, without dissipation (denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively) interconnected to each other via a Stokes-Dirac structure, also without dissipation (denoted $\mathcal{D}_\infty$). We consider here the simple case $p = q = n = 1$ throughout, for the Stokes-Dirac structure (though it can be extended, if not easily, to the higher dimensional case).

First we consider the composition of the two Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_\infty$. Consider $\mathcal{D}_1$ on the product space $\mathcal{F}_1 \times \mathcal{F}_0$ of two linear spaces $\mathcal{F}_1$ and $\mathcal{F}_0$, and the Stokes-Dirac structure $\mathcal{D}_\infty$ on the product space $\mathcal{F}_0 \times \mathcal{F}_{p,q} \times \mathcal{F}_l$. $\mathcal{F}_0$ and $\mathcal{F}_l$ are linear spaces (which represent the space of boundary variables of

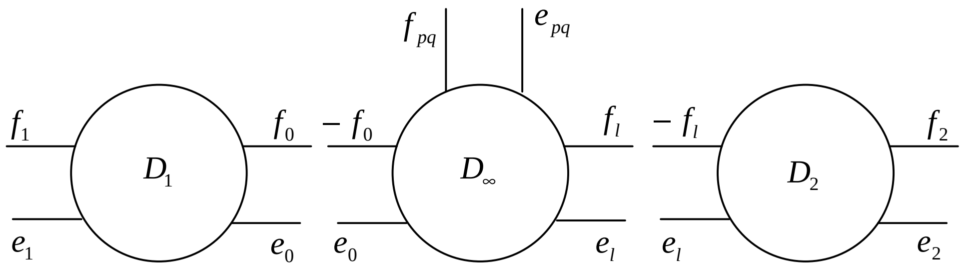


Figure 3.5: $\mathcal{D}_1 \parallel \mathcal{D}_\infty \parallel \mathcal{D}_2$

the Stokes-Dirac structure) and $\mathcal{F}_{p,q}$ is an infinite-dimensional function space with p, q representing the two different physical energy domains interacting with each other. The linear space $\mathcal{F}_0$ is the space of shared flow variables and its dual $\mathcal{F}_0^*$, the space of shared effort variables between $\mathcal{D}_1$ and $\mathcal{D}_\infty$. Next consider the composition of $\mathcal{D}_\infty$ and $\mathcal{D}_2$. Considering $\mathcal{D}_2$ as defined on the product space $\mathcal{F}_l \times \mathcal{F}_2$ of two linear spaces, the linear space $\mathcal{F}_l$ is the space of shared flow variables and its dual $\mathcal{F}_l^*$ is the space of shared effort variables between $\mathcal{D}_2$ and $\mathcal{D}_\infty$.

We define the two interconnections as follows: The interconnection of the two Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_\infty$ is defined as

$$\begin{aligned}\mathcal{D}_1 \| \mathcal{D}_\infty := \quad & \{f_1, e_1, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^* \mid \\ & \exists (f_0, e_0) \in \mathcal{F}_0 \times \mathcal{F}_0^* \text{ s.t} \\ & (f_1, e_1, f_0 . e_0) \in \mathcal{D}_1 \text{ and } (-f_0, e_0, f_p, f_q . e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty\}.\end{aligned}$$

Similarly, the interconnection of $\mathcal{D}_\infty$ and $\mathcal{D}_2$ is defined as

$$\begin{aligned}\mathcal{D}_\infty \| \mathcal{D}_2 := \quad & \{-f_0, e_0, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid \\ & \exists (f_l, e_l) \in \mathcal{F}_l \times \mathcal{F}_l^* \text{ s.t} \\ & (-f_0, e_0, f_p, f_q . e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \text{ and } (-f_l, e_l, f_2, e_2) \in \mathcal{D}_2\}.\end{aligned}$$

Hence we can define the total interconnection of $\mathcal{D}_1$,$\mathcal{D}_\infty$ and $\mathcal{D}_2$ as (also see Figure(3.5))

$$\begin{aligned}& \mathcal{D}_1 \| \mathcal{D}_\infty \| \ \mathcal{D}_2 := \{(f_1, e_1, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid \\ & \exists (f_0, e_0) \in \mathcal{F}_0 \times \mathcal{F}_0^* \text{ s.t } (f_1, e_1, f_0 . e_0) \in \mathcal{D}_1 \ ,(-f_0, e_0, f_p, f_q . e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty, \\ & \exists (f_l, e_l) \in \mathcal{F}_l \times \mathcal{F}_l^* \text{ s.t } (-f_0, e_0, f_p, f_q . e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty, \ (-f_l, e_l, f_2 . e_2) \in \mathcal{D}_2\}.\end{aligned} \tag{3.20}$$

This yields the following bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$:

$$\begin{aligned}
&\ll (f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a), (f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \gg \\
&:= < e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_2^a | f_2^b > + < e_2^b | f_2^a > \\
&+ \int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right]. \qquad (3.21)
\end{aligned}$$

Theorem 3.15. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_\infty$ be Dirac structures as said above (defined respectively with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$, $\mathcal{F}_l \times \mathcal{F}_l^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$). Then $D = \mathcal{D}_1 \| \mathcal{D}_\infty \| \mathcal{D}_2$ is a Dirac structure defined with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ given by (3.21).*

We use the following facts for the proof (as we know that $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_\infty$ individually are Dirac structures). On $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$ the bilinear form is defined as

$$\begin{aligned}
&\ll (f_1^a, f_0^a, e_1^a, e_0^a), (f_1^b, f_0^b, e_1^b, e_0^b) \gg \\
&:= < e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_0^b | f_0^a > + < e_0^a | f_0^b > . \qquad (3.22)
\end{aligned}$$

and $\mathcal{D}_1 = \mathcal{D}_1^\perp$ with respect to the bilinear form as in (3.22).
Similarly on $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_l \times \mathcal{F}_l^*$ the bilinear form is defined as

$$\begin{aligned}
&\ll (-f_l^a, e_l^a, f_2^a, e_2^a), (-f_l^b, e_l^b, f_2^b, e_2^b) \gg \\
&:= < e_2^b | f_2^a > + < e_2^a | f_2^b > - < e_l^b | f_l^a > - < e_l^a | f_l^b > \qquad (3.23)
\end{aligned}$$

and $\mathcal{D}_2 = \mathcal{D}_2^\perp$ with respect to the bilinear form as in (3.23).
On $\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$ the bilinear form takes the following form

$$\begin{aligned}
&\ll (f_p^a, f_q^a, f_b^a, e_p^a, e_q^a, e_b^a), (f_p^b, f_q^b, f_b^b, e_p^b, e_q^b, e_b^b) \gg \\
&:= \int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] + \\
&\left[< e_l^a | f_l^b > + < e_l^b | f_l^a > - < e_0^a | f_0^b > + < e_0^b | f_0^a > \right] \qquad (3.24)
\end{aligned}$$

and $\mathcal{D}_\infty = \mathcal{D}_\infty^\perp$ with respect to the bilinear form as in (3.24).

Proof. (i) $\mathcal{D} \subset \mathcal{D}^\perp$: Let $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}$ and consider any other $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$ and the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ as in (3.21). Then
$\exists\ (f_0^a, e_0^a)$, (f_l^a, e_l^a) $s.t$ $(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathcal{D}_1$, $(-f_0^a, e_0^a, f_p^a, f_q^a, e_p^a, e_q^a, f_l^a, e_l^a) \in \mathcal{D}_\infty$ and $(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathcal{D}_2$ and
$\exists\ (f_0^b, e_0^b)$, (f_l^b, e_l^b) $s.t$ $(f_1^b, e_1^b, f_0^b, e_0^b) \in \mathcal{D}_1$, $(-f_0^b, e_0^b, f_p^b, f_q^b, e_p^b, e_q^b, f_l^b, e_l^b) \in \mathcal{D}_\infty$

and $(-f_l^b, e_l^b, f_2^b, e_2^b) \in \mathcal{D}_2$.
Since $\mathcal{D}_\infty$ is a Dirac structure with respect to (3.24)

$$\begin{aligned}\int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b\right] = \\ - < e_l^a | f_l^b > - < e_l^b | f_l^a > + < e_0^a | f_0^b > + < e_0^b | f_0^a > .\end{aligned} \tag{3.25}$$

Substituting (3.25) in (3.21) and using the fact that the bilinear from (3.22) is zero on $\mathcal{D}_1$ and (3.23) is zero on $\mathcal{D}_2$, we get

$$\begin{aligned}&< e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_2^b | f_2^a > + < e_2^a | f_2^b > \\ &+ \int_z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b\right] = 0.\end{aligned}$$

and hence $\mathcal{D} \subset \mathcal{D}^\perp$
(ii) $\mathcal{D}^\perp \subset \mathcal{D}$: We know that the flow and effort variables of $\mathcal{D}_\infty$ are related as

$$D_\infty \triangleq \left\{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_p|\partial Z \\ e_q|\partial Z \end{bmatrix} \right\}. \tag{3.26}$$

Let $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}^\perp$, then for all $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$ the right side of Equation (3.21) is zero.
Now consider the vectors $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$ with $f_1^b = f_2^b = e_1^b = e_2^b = 0$ and also $f_0^b = e_0^b = f_l^b = e_l^b = 0$. Then from (3.26) and (3.21) we have

$$\int_Z \left[e_p^a \wedge \mathrm{d}e_q^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge \mathrm{d}e_p^b\right] = 0. \tag{3.27}$$

This implies (see the proof of Theorem 2.1 in [61])

$$f_p^a = \mathrm{d}e_q^a \text{ and } f_q^a = \mathrm{d}e_p^a. \tag{3.28}$$

Substituting (3.28) in (3.21) we have

$$\begin{aligned}&< e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_2^b | f_2^a > + < e_2^a | f_2^b > + \\ &\textstyle\int_Z \left[e_p^a \wedge \mathrm{d}e_q^b + e_p^b \wedge \mathrm{d}e_q^a + e_q^b \wedge \mathrm{d}e_p^a + e_q^a \wedge \mathrm{d}e_p^b\right] = 0.\end{aligned}$$

This yields by Stokes' theorem

$$\begin{aligned}&< e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_2^b | f_2^a > + < e_2^a | f_2^b > \\ &\qquad + \left[< e_p^a | e_q^b > + < e_p^b | e_q^a >\right] |_0^l = 0,\end{aligned}$$

for all e_p, e_q. Expanding the above and substituting for the boundary conditions we get

$$\begin{aligned}&< e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_0^a \mid f_0^b > + < e_0^b \mid f_0^a > + \\ &< e_2^b | f_2^a > + < e_2^a | f_2^b > - < e_l^a \mid f_l^b > - < e_l^b \mid f_l^a > = 0.\end{aligned} \tag{3.29}$$

Since $(f_0^b, e_0^b, f_l^b, e_l^b)$ are arbitrary and with $f_l^b = e_l^b = f_2^b = e_2^b = 0$ the above equation reduces to

$$< e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_0^a \mid f_0^b > + < e_0^b \mid f_0^a >= 0,$$

which implies that $(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathcal{D}_1$.
With similar arguments (with $f_l^b = e_l^b = f_1^b = e_1^b = 0$) Equation (3.29) reduces to

$$< e_2^b | f_2^a > + < e_2^a | f_2^b > - < e_l^a \mid f_l^b > - < e_l^b \mid f_l^a >= 0,$$

implying $(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathcal{D}_2$ and hence $\mathcal{D}^\perp \subset \mathcal{D}$, completing the proof.

Remark 3.16. Similarly we can also study interconnections of mixed finite and infinite-dimensional systems where we also have dissipation in the respective subsystems, which would again result in a port-Hamiltonian system with dissipation. The following corollary answers this problem.

Corollary 3.17. *Let $\mathcal{D}_1 \parallel \mathcal{R}_1$, $\mathcal{D}_2 \parallel \mathcal{R}_2$ and $\mathcal{D}_\infty \parallel \mathcal{R}_\infty$ be Dirac structures as defined above interconnected to their respective resistive relations, then the composed system will again have a structure of the form $\mathcal{D} \parallel \mathcal{R}$ with the property that $(\mathcal{D} \parallel \mathcal{R})^\perp = \mathcal{D} \parallel -\mathcal{R}$ where $-\mathcal{R}$ is a pseudo resistive relation (corresponding to negative resistance). Here $\mathcal{D}$ is the composition of the individual Dirac structures and $\mathcal{R}$ is the composition of the individual resistances of the subsystems.*

□

Example 3.18. We consider a port-Hamiltonian plant described by

$$\begin{aligned} f_p &= -[J(x) - R(x)]e_p - g(x)f \\ e &= g^T(x)e_p, \end{aligned} \tag{3.30}$$

interconnected to a port-Hamiltonian controller described by

$$\begin{aligned} f_c &= -[J_c(\xi) - R_c(\xi)]e_c - g_c(\xi)f' \\ e' &= g_c^T(\xi)e_c, \end{aligned}$$

through a transmission line which is an infinite-dimensional system described by (2.55). The interconnection constraints are of the form (see Figure 3.6)

$$\begin{aligned} e' &= y_c = f_0, \quad f = u_c = e_0, \\ e &= y_p = e_l, \quad\ f' = u_p = -f_l. \end{aligned} \tag{3.31}$$

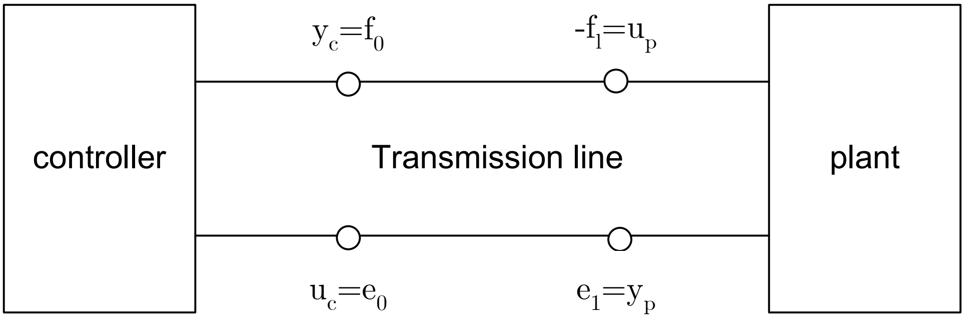


Figure 3.6: Interconnection constraints

With the above interconnection constraints the closed-loop dynamics can be written as

$$\begin{bmatrix} f_p \\ f_c \\ f_q \\ f_\phi \end{bmatrix} = \begin{bmatrix} -[J(x)-R(x)] & 0 & 0 & -g(x)\cdot\mid_l \\ 0 & -[J_c(\xi)-R_c(\xi)] & -g_c(\xi)\cdot\mid_0 & 0 \\ 0 & 0 & 0 & \mathrm{d} \\ 0 & 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_c \\ e_q \\ e_\phi \end{bmatrix}$$

$$\begin{bmatrix} e_{ql} \\ e_{\phi 0} \end{bmatrix} = \begin{bmatrix} g^T(x)e_p \\ -g_c(\xi)e_c \end{bmatrix}. \tag{3.32}$$

In energy variables the overall dynamics is given as

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \\ \frac{\partial}{\partial t}q(z,t) \\ \frac{\partial}{\partial t}\phi(z,t) \end{bmatrix} = \begin{bmatrix} [J(x)-R(x)] & 0 & 0 & -g(x)\cdot\mid_l \\ 0 & [J_c(\xi)-R_c(\xi)] & g_c(\xi)\cdot\mid_0 & 0 \\ 0 & 0 & 0 & \mathrm{d} \\ 0 & 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}H(x) \\ \frac{\partial}{\partial \xi}H(\xi) \\ \delta_q\mathcal{H}(\bar{q}) \\ \delta_\phi\mathcal{H}(\bar{q}) \end{bmatrix}$$

$$\begin{bmatrix} \delta_q\mathcal{H}(\bar{q})\mid_l \\ \delta_\phi\mathcal{H}(\bar{q})\mid_0 \end{bmatrix} = \begin{bmatrix} g^T(x)\frac{\partial}{\partial x}H(x) \\ -g_c(\xi)\frac{\partial}{\partial \xi}H(\xi) \end{bmatrix}. \tag{3.33}$$

It immediately follows that the above system is in the port-Hamiltonian form. The closed-loop energy defined in the extended state space $\chi = [x, \xi, q_E(z,t), q_M(z,t)]^T$ is given by

$$H_{cl}(\chi) = H(x) + H_c(\xi) + \mathcal{H}(\bar{q}),$$

with energy rate

$$\dot{H}_{cl} = -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial^T H}{\partial x}(x) - \frac{\partial^T H_c}{\partial \xi}(\xi)R_c(\xi)\frac{\partial H_c}{\partial \xi}(\xi).$$

Interconnections in a higher dimensional case

In the previous section on the interconnection of infinite-dimensional systems to finite-dimensional systems through the boundary of the infinite-dimensional system, we considered the simple case where $p = q = n = 1$ for the infinite-dimensional system. This corresponds to the case of a distributed system with a 1-D spatial domain. In this section we highlight briefly on how this could be generalized to an infinite-dimensional system with an n-dimensional spatial domain and how these systems could be interconnected through the boundary to finite-dimensional systems.

Consider the dynamics of then 2-D shallow water equations which are given by

$$\partial_t \tilde{h} + \partial_{z_1}(\tilde{h}\tilde{u}) + \partial_{z_2}(\tilde{h}\tilde{v}) = 0$$

$$\partial_t \tilde{u} + \partial_{z_1}(\tfrac{1}{2}\tilde{u}^2 + g\tilde{h}) + \tilde{v}\partial_{z_2}\tilde{u} = 0$$

$$\partial_t \tilde{v} + \tilde{u}\partial_{z_1}\tilde{v} + \partial_{z_2}(\tfrac{1}{2}\tilde{v}^2 + g\tilde{h}) = 0.$$

The formulation of the above equations as a port-Hamiltonian system is given as follows. Let $W \subset \mathbb{R}^2$ be a given domain over which the water flows. We assume the existence of a Riemannian metric $< \cdot >$ on W, usually the standard Euclidean metric on $\mathbb{R}^2$. Let $Z \subset W$ be any two-dimensional manifold with boundary ∂Z. We identify the height $h(z_1, z_2, t) = \tilde{h}(z_1, z_2, t)dz_1 dz_2$ (which represents the mass density) with a two-form on Z, that is with elements in $\Omega^2(Z)$. Furthermore we identify the Eulerian vector field $V(z_1, z_2, t)$ with a one-form on Z, that is, with an element in $\Omega^1(Z)$, i.e. $V(z_1, z_2, t) = u(z_1, z_2, t)dz_1 + v(z_1, z_2, t)dz_2$. The spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ are given by

$$\begin{aligned}\mathcal{F}_{p,q} &= \Omega^2(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z)\\ \mathcal{E}_{p,q} &= \Omega^0(Z) \times \Omega^1(Z) \times \Omega^1(\partial Z),\end{aligned}$$

with the corresponding Stokes-Dirac structure $\mathcal{D}$ on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$. We now have the following modified Stokes-Dirac structure

$$\begin{aligned}&\mathcal{D} := \{(f_h, f_V, f_b, e_h, e_V, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid \\ &\begin{bmatrix} f_h \\ f_V \end{bmatrix} = \begin{bmatrix} \mathrm{d}e_V \\ \mathrm{d}e_h + \frac{1}{*h}((*\mathrm{d}V) \wedge (*e_V)) \end{bmatrix}; \begin{bmatrix} f_b \\ e_b \end{bmatrix} \begin{bmatrix} e_h \mid_{\partial Z} \\ -e_V \mid_{\partial Z} \end{bmatrix} \}.\end{aligned}$$

In terms of the $2 - d$ shallow water equations would corresponds to

$$\begin{aligned}f_h &= -\frac{\partial}{\partial t} h(z,t), e_h = \delta_h H = \frac{1}{2}(< V^\sharp, V^\sharp > + g(*h))\\ f_V &= -\frac{\partial}{\partial t} V(z,t), e_v = \delta_V H = (*h)(*V),\end{aligned}$$

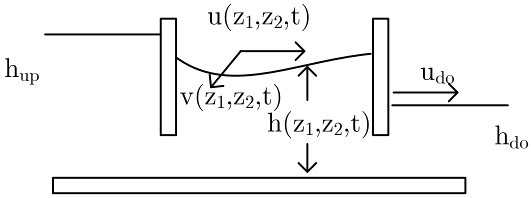


Figure 3.7: The 2-D water flow

together with the boundary variables $f_b = \delta_h H \mid_{\partial Z}$ called the Bernoulli function and $e_b = \delta_V H \mid_{\partial Z}$ denoting the boundary mass flow.

Consider the interconnection of this infinite-dimensional system, with finite-dimensional systems through its boundaries, see Figure 3.7. The finite-dimensional systems can be thought of water reservoirs given in port-Hamiltonian form as

$$\begin{aligned} \dot{x}_i &= u_i \\ y_i &= \frac{\partial H_i(x_i)}{\partial x_i}; \quad i = 1, 2. \end{aligned}$$

The interconnection constraints at the gates would then be as follows

$$\begin{aligned} &u_0 = \textstyle\int e_{b0} = \int (*h * V) \mid_0, && u_1 = \textstyle\int e_{b1} = \int (*h * V) \mid_1, \\ &y_0 = -f_0 = \tfrac{1}{2}(< V^\sharp, V^\sharp > + g(*h)) \mid_0, && y_1 = -f_1 = \tfrac{1}{2}(< V^\sharp, V^\sharp > + g(*h)) \mid_1 . \end{aligned}$$

It can easily be seen that such interconnection constraints are indeed power-conserving and the total interconnection is again a Dirac structure. Using similar arguments we can generalize this to a infinite-dimensional system with a $n-$dimensional spatial domain.

Consider an infinite-dimensional port-Hamiltonian system, defined with respect to a Stokes-Dirac structure (2.46) defined on the product space $\mathcal{F}_{p,q} \times \mathcal{F}_b$ with port variables $(f_p, f_q, e_p, e_q, f_b, e_b)$. Now consider interconnection of this system with a finite-dimensional Dirac structure through the boundary. The finite dimensional Dirac structure is defined on the product space $\mathcal{F}_1 \times \mathcal{F}_b'$ with port-variables (f_1, e_1, f_b', e_b'). The following interconnection constraints then define an interface between the boundary variables of the infinite-dimensional system and the port variables of the finite-dimensional system which are available for interconnection.

$$\begin{aligned} e_b' &= \textstyle\int_{\partial Z} e_b \\ f_b' &= f_b. \end{aligned} \tag{3.34}$$

It can easily be seen that the interconnected system is again a port-Hamiltonian system and the composed Dirac structure is defined on the product space $\mathcal{F}_{p,q} \times \mathcal{F}_1$.

3.3.2 Interconnections of infinite-dimensional systems through a distributed finite-dimensional system

So far we have studied interconnections of different types of systems namely, interconnection of finite-dimensional systems with finite-dimensional systems and interconnections of infinite-dimensional systems with infinite-dimensional systems either through the spatial domain or through the boundary. We have also seen interconnection of infinite-dimensional systems with an $n-$dimensional spatial domain thorough the boundary with finite-dimensional systems, the case which we call a mixed port-Hamiltonian system. We now study a case of two infinite-dimensional systems interconnected to each other through a finite-dimensional system and this interconnection takes place through the spatial domains of the infinite-dimensional systems. Towards the end we also present a simple example to further motivate this case.

Let $\mathcal{D}_{\infty_1}$ be a Stokes-Dirac structure which is defined on the product space $\mathcal{F}_{p,q} \times \mathcal{F}_b \times \mathcal{F}_1$, with the elements of the Dirac structure being $(f_{p_1}, f_{q_1}, e_{p_1}, e_{q_1}, f_{b_1}, e_{b_1}, f_1, e_1)$, the variables (f_1, e_1) corresponding to the flows and efforts of an open port in the spatial domain of the system. Similarly consider another Stokes-Dirac structure denoted by $\mathcal{D}_{\infty_2}$ defined on the product space $\mathcal{F}'_{p,q} \times \mathcal{F}'_b \times \mathcal{F}_2$, the elements of the Dirac structure being $(f_{p_2}, f_{q_2}, e_{p_2}, e_{q_2}, f_{b_2}, e_{b_2}, f_2, e_2)$ again (f_2, e_2) corresponding to the flow and effort arising due to an open port in the spatial domain of $\mathcal{D}_{\infty_2}$. Lastly, let $\mathcal{D}$ be a finite-dimensional Dirac structure defined on the space $\mathcal{F}_s \times \mathcal{E}_s \times \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$, with the elements of the Dirac structure being $(f_s, e_s, f'_1, e'_1, f'_2, e'_2)$, with (f_s, e_s) corresponding to the flow and effort variables of the energy storing elements and (f'_1, e'_1, f'_2, e'_2) corresponding to the ports available for interconnection. The space $\mathcal{F}_1 \times \mathcal{E}_1$ is the space of shared flow and effort variables between $\mathcal{D}_{\infty_1}$ and $\mathcal{D}$ and the space $\mathcal{F}_2 \times \mathcal{E}_2$ is the space of shared flow and effort variables between $\mathcal{D}_{\infty_2}$ and $\mathcal{D}$

We now define the interconnection constraints between $\mathcal{D}_{\infty_1}$ and $\mathcal{D}$ as follows

$$
\begin{aligned}
\mathcal{D}_{\infty_1} \parallel \mathcal{D} := \quad & \{f_{p_1}, f_{q_1}, e_{p_1}, e_{q_1}, f_{b_1}, e_{b_1}, f_s, e_s) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b \times \mathcal{F}_s \times \mathcal{E}_s | \\
& \exists (f_1, e_1) \in \mathcal{F}_1 \times \mathcal{E}_1 \text{ s.t.} \\
& (f_{p_1}, f_{q_1}, e_{p_1}, e_{q_1}, f_{b_1}, e_{b_1}, f_1, e_1) \in \mathcal{D}_{\infty_1}, (f_q, e_q, -f_1, e_1, f_2, e_2) \in \mathcal{D}\}.
\end{aligned}
$$

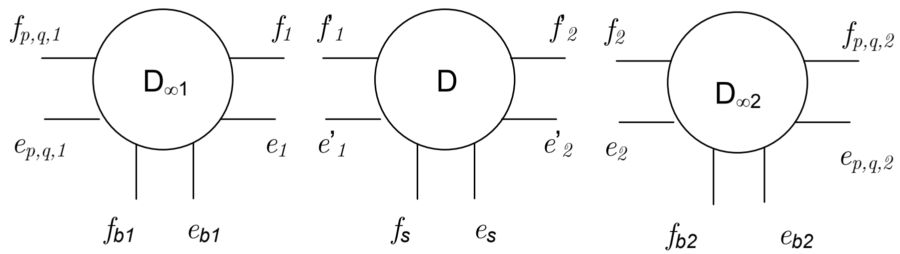


Figure 3.8: $\mathcal{D}_{\infty_1} \parallel \mathcal{D} \parallel \mathcal{D}_{\infty_2}$

Similarly we define the interconnection between $\mathcal{D}_{\infty_2}$ and $\mathcal{D}$ as follows

$$\begin{aligned}\mathcal{D}_{\infty_2} \parallel \mathcal{D} := \quad & \{f_{p_2}, f_{q_2}, e_{p_2}, e_{q_2}, f_{b_2}, e_{b_2}, f_s, e_s) \in \mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q} \times \mathcal{F}'_b \times \mathcal{E}'_b \times \mathcal{F}_s \times \mathcal{E}_s \mid \\ & \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2 \text{ s.t.} \\ & (f_{p_2}, f_{q_2}, e_{p_2}, e_{q_2}, f_{b_2}, e_{b_2}, f_2, e_2) \in \mathcal{D}_{\infty_2}, (f_s, e_s, f_1, e_1, -f_2, e_2) \in \mathcal{D}.\end{aligned}$$

We can then define the total interconnection as follows

$$\begin{aligned}\mathcal{D}_{\infty_1} \parallel \mathcal{D} \parallel \mathcal{D}_{\infty_2} := \quad & \{(f_{p_1}, f_{q_1}, e_{p_1}, e_{q_1}, f_{b_1}, e_{b_1}, f_s, e_s f_{p_2}, f_{q_2}, e_{p_2}, e_{q_2}, f_{b_2}, e_{b_2}) \\ & \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b \times \mathcal{F}_s \times \mathcal{E}_s \times \mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q} \times \mathcal{F}'_b \times \mathcal{E}'_b \mid \\ & \exists (f_1, e_1) \in \mathcal{F}_1 \times \mathcal{E}_1 \text{ s.t.} (f_{p_1}, f_{q_1}, e_{p_1}, e_{q_1}, f_{b_1}, e_{b_1}, f_1, e_1) \in \mathcal{D}_{\infty_1} \\ \text{and} \quad & (f_q, e_q, -f_1, e_1, f_2, e_2) \in \mathcal{D}; \\ \text{and} \quad & \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2 \text{ s.t. } (f_{p_2}, f_{q_2}, e_{p_2}, e_{q_2}, f_{b_2}, e_{b_2}, f_2, e_2) \in \mathcal{D}_{\infty_2} \\ \text{and} \quad & (f_s, e_s, f_1, e_1, -f_2, e_2) \in \mathcal{D}\}.\end{aligned}$$

This yields a bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b \times \mathcal{F}_s \times \mathcal{E}_s \times \mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q} \times \mathcal{F}'_b \times \mathcal{E}'_b$:

$$\begin{aligned}& << (f^a_{p_1}, f^a_{q_1}, e^a_{p_1}, e^a_{q_1}, f^a_{b_1}, e^a_{b_1}, f^a_s, e^a_s, f^a_{p_2}, f^a_{q_2}, e^a_{p_2}, e^a_{q_2}, f^a_{b_2}, e^a_{b_2}), \\ & (f^b_{p_1}, f^b_{q_1}, e^b_{p_1}, e^b_{q_1}, f^b_{b_1}, e^b_{b_1}, f^b_s, e^b_s f^b_{p_2}, f^b_{q_2}, e^b_{p_2}, e^b_{q_2}, f^b_{b_2}, e^b_{b_2}) >> \\ & := \int_{Z_1} [e^b_{p_1} \wedge f^a_{p_1} + e^a_{p_1} \wedge f^b_{p_1} + e^b_{q_1} \wedge f^a_{q_1} + e^a_{q_1} \wedge f^b_{q_1}] + \\ & \quad \int_{Z_2} [e^b_{p_2} \wedge f^a_{p_2} + e^a_{p_2} \wedge f^b_{p_2} + e^b_{q_2} \wedge f^a_{q_2} + e^a_{q_2} \wedge f^b_{q_2}] \\ & + \int_{\partial Z_1} [e^b_{b_1} \wedge f^a_{b_1} + e^a_{b_1} \wedge f^b_{b_1}] + \int_{\partial Z_2} [e^b_{b_2} \wedge f^a_{b2} + e^a_{b_2} \wedge f^b_{b_2}] \\ & + < e^b_s \mid f^a_s > + < e^a_s \mid f^b_s > .\end{aligned} \tag{3.35}$$

We then state the following proposition:

Proposition 3.19. *Let $\mathcal{D}_{\infty_1}$, $\mathcal{D}_{\infty_2}$ and $\mathcal{D}$ be Dirac structures as stated above, defined respectively with respect to $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b \times \mathcal{F}_1 \times \mathcal{E}_1$, $\mathcal{F}'_{p,q} \times \mathcal{E}'_{p,q} \times$*

$\mathcal{F}_b' \times \mathcal{E}_b' \times \mathcal{F}_2 \times \mathcal{E}_2$ and $\mathcal{F}_s \times \mathcal{E}_s \times \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$. The $\mathcal{D} = \mathcal{D}_{\infty_1} \parallel \mathcal{D} \parallel \mathcal{D}_{\infty_2}$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_b \times \mathcal{E}_b \times \mathcal{F}_s \times \mathcal{E}_s \times \mathcal{F}_{p,q}' \times \mathcal{E}_{p,q}' \times \mathcal{F}_b' \times \mathcal{E}_b'$ given by (3.35)

Proof. The proof follows the same arguments as in the proof of the Theorem 3.15 and hence is omitted. □

To illustrate the above interconnection we present an example here of connecting two vibrating strings in parallel through a *distributed* spring.

Example: The Coupled wave equations

Consider two strings in parallel connected through a distributed spring. The two strings are modeled by the $1-d$ wave equation

$$\mu_i \ddot{u}_i + E_i \Delta u_i = 0, \ i = 1, 2$$

where μ_i is the mass density of each string and E_i is the Young's modulus. This equation models the vertical movement $u(z,t)$ of the vibrating membrane. The port-Hamiltonian model of each of the system with external forces through the spatial domain is given as

$$\begin{bmatrix} f_{\epsilon_i} \\ f_{\rho_i} \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_{\epsilon i} \\ e_{\rho_i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_i(z,t)$$
$$E_i = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}_i}{\partial \epsilon_i} \\ \frac{\partial \mathcal{H}_i}{\partial \rho_i} \end{bmatrix}; \begin{bmatrix} v_{bi} \\ \sigma_{bi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_{\epsilon i} \\ e_{\rho_i} \end{bmatrix} \Bigg|_{\partial Z}.$$

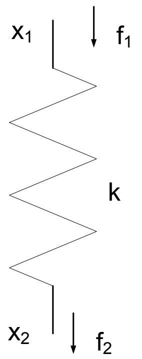
The energy variables are the $1-$form kinetic momentum $\rho_i(z,t)$ and the $1-$ form elastic strain $\epsilon_i(z,t)(= \frac{\partial u_i}{\partial z} dz)$. The flows are then given by $f_{\epsilon_i} = \dot{\epsilon}_i$ and $f_{\rho_i} = \dot{\rho}_i$. The co-energy variables are then the $0-$form velocity $v_i(z,t) = \frac{\partial H_i}{\partial \rho_i}$ denoted by e_{ρ_i} and the $0-$ form stress $\sigma_i(z,) = \frac{\partial H_i}{\partial \epsilon_i}$, which is denoted by $e_{\epsilon i}$. $F_i(z,t)$ is the external force acting on the spring through the spatial domain of the system, E_i the corresponding output (the velocity), H_i is the Hamiltonian density defined as

$$H_i(\rho_i, \epsilon_i) = \frac{1}{2}(\epsilon_i \wedge \sigma_i + \rho_i \wedge v_i),$$

where $\wedge$ is the wedge product of differential forms, the co-energy variables σ and v are related to the energy variables by the constitutive relations

$$\sigma_i = E_i * \epsilon_i, \quad v_i = \tfrac{1}{\mu_i} * \rho_i \ .$$

The port-Hamiltonian model of the spring is given as follows. Let $q = x_1 - x_2$ be the elongation of the spring. At equilibrium, the sum of the forces is zero,



i.e., $f_1 + f_2 = 0$. The force and the displacement relationship is given by $f_1 = k(x_1 - x_2)$, where k is the spring constant and the potential energy of the spring is given by

$$H_s = \frac{1}{2}kq^2,$$

writing this in the port-Hamiltonian form, i.e. $\dot{x} = J\frac{\partial H}{\partial x} + gu$ and $y = g^T\frac{\partial H}{\partial x}$, gives

$$\dot{q} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} q.$$

where $J = 0$, $g = [1\ {-1}]$, $y = [f_1\ f_2]^T\ \frac{\partial H}{\partial x} = [kq\ \ {-kq}]^T$ and $u = [v_1\ v_2]$.

Next we study the interconnections of the two strings through the spring and see that the resulting system is again a port-Hamiltonian system. The distributed interconnection constraints are as follows

$$F_i = -f_i, \quad E_i = v_i, \quad i = 1, 2\ .$$

The composed system can then be written as follows

$$\begin{bmatrix} f_{\epsilon_1} \\ f_{\rho_1} \\ f_s \\ f_{\epsilon_2} \\ f_{\rho_2} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} & 0 & 0 & 0 \\ -\mathrm{d} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \mathrm{d} \\ 0 & 0 & 0 & -\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_{\epsilon_1} \\ e_{\rho_1} \\ e_s \\ e_{\epsilon_2} \\ e_{\rho_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_{\epsilon_1} \\ e_{\rho_1} \\ e_s \\ e_{\epsilon_2} \\ e_{\rho_2} \end{bmatrix}, \tag{3.36}$$

together with the boundary conditions on the two strings given by

$$\begin{bmatrix} v_{bi} \\ \sigma_{bi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_i}{\partial \epsilon} \\ \frac{\partial H_i}{\partial \rho} \end{bmatrix} ; \; i = 1, 2.$$

It can easily be seen that the interconnected system is again a port-Hamiltonian system, with the overall dynamics of the composed system given by the above equations. The total energy of the system is given by

$$H = \frac{1}{2} \int_Z \left[(\epsilon_1 \wedge \sigma_1 + \rho_1 \wedge v_1) + (\epsilon_2 \wedge \sigma_2 + \rho_2 \wedge v_2) + kq^2 \right].$$

4

Casimirs and its Implications on Control

"The greatest challenge to any thinker is stating the problem in a way that will allow a solution." - Bertrand Russell.

In the previous chapter we have seen how a power-conserving interconnection of two or more Dirac structures with partially shared variables is again a Dirac structure. This in turn means that a power-conserving interconnection of a number of port-Hamiltonian system is again a port-Hamiltonian system. The Dirac structure of the composed system is the composition of Dirac structures of the constituent parts, Hamiltonian being the sum of individual Hamiltonians and the total resistive relation being the composition of the individual resistive relations. In this chapter we focus on how to exploit these properties for further analysis of port-Hamiltonian systems.

In particular, we investigate which closed-loop port-Hamiltonian systems can be achieved by interconnecting a given plant port-Hamiltonian system P with a to-be-designed controller port-Hamiltonian system C. This is quite important from the point of view of the theory of control by interconnection for stabilizing port-Hamiltonian systems, which relies on the generation of conserved quantities, called Casimirs, for the closed-loop system. In this chapter we address the question of achievable Dirac structures for finite-dimensional systems, infinite-dimensional systems defined with respect to a Stokes-Dirac structure and also the case of mixed finite and infinite-dimensional systems. We also characterize the set of achievable Casimirs for the closed-loop system. In the case of finite-dimensional systems we see that this characterization of the set of achievable Casimirs in terms of plant state enables us to determine, without a priori knowledge of the controller system, whether or not there exist Casimirs for the closed-loop system and hence the applicability of the control by interconnection (or the Energy-Casimir) method. We also focus on the precise role of energy dissipation in the analysis of port-Hamiltonian systems and in the case of finite-dimensional systems with dissipation we see that, under certain conditions, if a function is a Casimir for a given resistive relation it is a Casimir for all resistive relations.

4.1 Casimirs

Casimirs are functions that are conserved quantities of the system for every Hamiltonian (see [61, 7, 58] for example), and they are completely characterized by the Dirac structures of the port-Hamiltonian systems. The existence of such functions has immediate consequences for stability analysis of systems. Suppose we want to stabilize a plant port-Hamiltonian system around a desired equilibrium x_*, and we would like to design a controller port-Hamiltonian system such that the closed-loop system has the desired stability properties. The closed-loop system then satisfies

$$\frac{d}{dt}(H_P + H_C) \leq 0.$$

In case x_* is not a minimum for H_p, then a possible strategy is that we generate Casimir functions $\mathcal{C}(x,\xi)$ for the closed-loop system by appropriately choosing the controller port-Hamiltonian system. The resulting Lyapunov function is then given by the sum of the plant and controller Hamiltonians and the corresponding Casimir function,

$$V(x,\xi) := H_P(x) + H_C(\xi) + \mathcal{C}(x,\xi),$$

where $H_P(x)$ is the plant Hamiltonian and $H_C(\xi)$ the controller Hamiltonian. The Lyapunov function should be constructed such that it has a minimum at (x_*, ξ_*), with ξ_* still to be chosen. This strategy is based on finding all the achievable Casimirs of the closed-loop system. Since the closed-loop Casimirs are based on the closed-loop Dirac structures, the problem reduces to finding all the closed-loop Dirac structures.

A Casimir function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ for a port-Hamiltonian system is a function which is constant along all the trajectories of the port-Hamiltonian system irrespective of the Hamiltonian. Consider, in the case of two external ports, the following subspace

$$G_1 := \{f \in \mathcal{F} \mid \exists e \in \mathcal{F}^* \text{ s.t } (f,e) \in \mathcal{D}\}.$$

A function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ is a Casimir function if $\frac{d\mathcal{C}}{dt}(x(t)) = \frac{\partial^T \mathcal{C}}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in G_1$. Hence $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ is a Casimir function for the port-Hamiltonian system if and only if

$$\frac{\partial \mathcal{C}}{\partial x}(x) \in G_1^{\perp}.$$

Geometrically this can be formulated by defining the following subspace of the dual space of efforts

$$P_0 = \{e \in \mathcal{F}^* \mid (0,e) \in \mathcal{D}\}.$$

It can easily be seen that $G_1^{\perp} = P_0$ where $\perp$ denotes the orthogonal complement with respect to the duality product $<|>$. Hence $\mathcal{C}$ is a Casimir function if and only if $\frac{d\mathcal{C}}{dt}(x) \in P_0$. In short we can say that a Casimir function for a port-Hamiltonian system is any function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ such that the Casimir gradients satisfy

$$(0, e) \in \mathcal{D}. \tag{4.1}$$

In case of a non-autonomous system, where now the elements of the Dirac structure are $(f, e, f', e') \in \mathcal{D}$, with (f', e') connected to the control ports, which are accessible for the controller interaction, a Casimir is a function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$, such that its gradient $e = \frac{\partial C}{\partial x}$ now satisfies

$$(0, e, f_c, e_c) \in \mathcal{D}, \tag{4.2}$$

for some f_c, e_c. This will imply that no longer $\frac{d\mathcal{C}}{dt} = 0$, but will depend on the variables at the control ports. Indeed, from (4.2) we have that

$$(0, \frac{\partial \mathcal{C}}{\partial x}, f_c, e_c) \in \mathcal{D} = \mathcal{D}^{\perp}.$$

This implies that

$$-\frac{\partial \mathcal{C}}{\partial x}\dot{x} + 0.e_s + f_c e' + e_c f' = 0,$$

for all $(-\dot{x}, e_s, f', e') \in \mathcal{D}$. This means that

$$\frac{d\mathcal{C}}{dt} = f_c e' + e_c f',$$

and hence $\frac{dC}{dt}$ is a linear function of f' and e'.

4.2 Achievable Casimirs for finite-dimensional systems

In this section we first investigate what are the achievable Dirac structures with dissipation of the closed-loop system. That is, given the Dirac structure with dissipation $\mathcal{D}_P \parallel \mathcal{R}_P$ of the plant system P and the to-be-designed Dirac structure with dissipation $\mathcal{D}_C \parallel \mathcal{R}_C$ of the controller system C, what are the achievable Dirac structures $(\mathcal{D}_P \parallel \mathcal{R}_P \parallel \mathcal{D}_C \parallel \mathcal{R}_C)$? Here $\parallel$ denotes the composition of various structures as defined in the previous chapter. To study this problem we first study the problem of achievable Dirac structures and similarly achievable resistive relations and then combine these two results.

4.2.1 Achievable Dirac structures

Proposition 4.1. *[7] Consider a (given) plant Dirac structure $\mathcal{D}_P$ with port variables (f_1, e_1, f_2, e_2), and a desired Dirac structure $\mathcal{D}$ with port variables (f_1, e_1, f_3, e_3).*

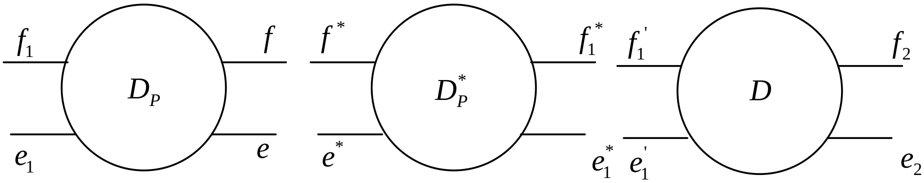


Figure 4.1: $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_P^* \parallel \mathcal{D}$

Then there exists a controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$ if and only if the following two equivalent conditions are satisfied

$$\mathcal{D}_P^0 \subset \mathcal{D}^0 \tag{4.3}$$

$$D^\pi \subset \mathcal{D}_P^\pi, \tag{4.4}$$

$$\textit{where} \begin{cases} \mathcal{D}_P^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}_P\} \\ \mathcal{D}_P^\pi := \{(f_1, e_1) \mid \exists (f, e) : (f_1, e_1, f, e) \in \mathcal{D}_P\} \\ \mathcal{D}^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}\} \\ \mathcal{D}^\pi := \{(f_1, e_1) \mid \exists (f_2, e_2) : (f_1, e_1, f_2, e_2) \in \mathcal{D}\}. \end{cases} \tag{4.5}$$

The following proof of Theorem 4.1 is based on the following 'copy' (or 'internal model') $\mathcal{D}_P^*$ of the plant Dirac structure $\mathcal{D}_P$:

$$\mathcal{D}_P^* := \{(f_1, e_1, f, e) \mid (-f_1, e_1, -f, e) \in \mathcal{D}_P\}. \tag{4.6}$$

It is easily seen that $\mathcal{D}_P^*$ is a Dirac structure if and only if $\mathcal{D}_P$ is a Dirac structure.

Proof of Proposition 4.1. Necessity of (4.3, 4.4) is obvious. Sufficiency is shown using the controller Dirac structure

$$\mathcal{D}_C := \mathcal{D}_P^* \parallel \mathcal{D},$$

(see Figure 4.1).

To check that $\mathcal{D} \subset \mathcal{D}_P \parallel \mathcal{D}_C$, consider $(f_1, e_1, f_2, e_2) \in \mathcal{D}$. Because $(f_1, e_1) \in \mathcal{D}^\pi$, applying (4.4) yields that $\exists (f, e)$ such that $(f_1, e_1, f, e) \in \mathcal{D}_P$. It follows that $(-f_1, e_1, -f, e) \in \mathcal{D}_P^*$. Recall now the following interconnection constraints in Figure 4.1

$$f = -f^*, e = e^*, f_1^* = -f_1', e_1^* = e_1'.$$

By taking $(f_1', e_1') = (f_1, e_1)$ in Figure 4.1 it follows that $(f_1, e_1, f_2, e_2) \in \mathcal{D}_P \parallel \mathcal{D}_C$. Therefore, $\mathcal{D} \subset \mathcal{D}_P \parallel \mathcal{D}_C$.

To check that $\mathcal{D}_P \parallel \mathcal{D}_C \subset \mathcal{D}$, consider $(f_1, e_1, f_2, e_2) \in \mathcal{D}_P \parallel \mathcal{D}_C$. Then there exist $f = -f^*, e = e^*$, $f_1^* = -f_1', e_1^* = e_1'$ such that

$$(f_1, e_1, f, e) \in \mathcal{D}_P \tag{4.7}$$

$$(f_1^*, e_1^*, f^*, e^*) \in \mathcal{D}_P^* \iff (f_1', e_1', f, e) \in \mathcal{D}_P \tag{4.8}$$

$$(f_1', e_1', f_2, e_2) \in \mathcal{D}. \tag{4.9}$$

Subtracting (4.8) from (4.7), making use of the linearity of $\mathcal{D}_P$, we get

$$(f_1 - f_1', e_1 - e_1', 0, 0) \in \mathcal{D}_P \iff (f_1 - f_1', e_1 - e_1') \in \mathcal{D}_P^0. \tag{4.10}$$

Using (4.10) and (4.3) we get

$$(f_1 - f_1', e_1 - e_1', 0, 0) \in \mathcal{D}. \tag{4.11}$$

Finally, adding (4.9) and (4.11), we obtain $(f_1, e_1, f_2, e_2) \in \mathcal{D}$, and so $\mathcal{D}_P \parallel \mathcal{D}_C \subset \mathcal{D}$.

Finally we show that conditions (4.3) and (4.4) are equivalent. In fact we prove that

$$(\mathcal{D}^0)^\perp = \mathcal{D}^\pi,$$

and the same for $\mathcal{D}_P$. Here, $^\perp$ denotes the orthogonal complement with respect to the canonical bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^*$ defined as

$$<< (f_1^a, e_1^a), (f_1^b, e_1^b) >>:=< e^a \mid f^b > + < e^b \mid f^a >,$$

for $(f_1^a, e_1^a), (f_1^b, e_1^b) \in \mathcal{F}_1 \times \mathcal{F}_1^*$. Then since $\mathcal{D}_P^0 \subset \mathcal{D}^0$ implies $(\mathcal{D}^0)^\perp \subset (\mathcal{D}_P^0)^\perp$. the equivalence between (4.3) and (4.4) is immediate.

In order to show $(\mathcal{D}^0)^\perp = \mathcal{D}^\pi$ first take $(f_1, e_1) \in (\mathcal{D}^\pi)^\perp$, implying that

$$<< (f_1, e_1), (\tilde{f}_1, \tilde{e}_1) >>=< e_1 \mid \tilde{f}_1 > + < \tilde{e}_1 \mid f_1 >= 0,$$

for all $(\tilde{f}_1, \tilde{e}_1)$ for which there exists $(\tilde{f}_2, \tilde{e}_2)$ such that $(\tilde{f}_1, \tilde{e}_1, \tilde{f}_2, \tilde{e}_2) \in \mathcal{D}$. This implies that $(f_1, e_1, 0, 0) \in \mathcal{D}^\perp = \mathcal{D}$ and thus that $(f_1, e_1) \in \mathcal{D}^0$. Hence, $(\mathcal{D}^\pi)^\perp \subset \mathcal{D}^0$ and thus $(\mathcal{D}^0)^\perp = \mathcal{D}^\pi$, implying that there exists (f_2, e_2) such that $(f_1, e_1, f_2, e_2) \in \mathcal{D} = \mathcal{D}^\perp$. Hence

$$< e_1 \mid \tilde{f}_1 > + < \tilde{e}_1 \mid f_1 > + < e_2 \mid \tilde{f}_2 > + < \tilde{e}_2 \mid f_2 >= 0,$$

for all $(\tilde{f}_1, \tilde{e}_1, \tilde{f}_2, \tilde{e}_2) \in \mathcal{D}$, implying that $< e_1 \mid \tilde{f}_1 > + < \tilde{e}_1 \mid f_1 >$ for all $(\tilde{f}_1, \tilde{e}_1, 0, 0) \in \mathcal{D}$ and thus $(f_1, e_1) \in (\mathcal{D}^0)^\perp$.

4.2.2 Achievable resistive relations

Similar analysis could also be done for resistive relation in which case we formulate the problem as follows: We are given a $\mathcal{R}_1$ and the to-be-designed $\mathcal{R}_2$, then what are the achievable resistive relations $\mathcal{R}_1 \parallel \mathcal{R}_2$?

Proposition 4.2. *Given a resistive relation $\mathcal{R}_1$ with port variables $(f_{R1}, e_{R1}, f_2, e_2)$ and a desired resistive relation $\mathcal{R}$ with port variables $(f_{R1}, e_{R1}, f_{R3}, e_{R3})$. Then there exists an $\mathcal{R}_2$ such that $\mathcal{R} = \mathcal{R}_1 \parallel \mathcal{R}_2$ if and only if the following two conditions are satisfied*

$$\mathcal{R}_1^0 \subset \mathcal{R}^0 \tag{4.12}$$
$$\mathcal{R}^\pi \subset \mathcal{R}_1^\pi, \tag{4.13}$$

where

$$\begin{aligned}
\mathcal{R}_1^0 &:= \{(f_{R1}, e_{R1}) \mid (f_{R1}, e_{R1}, 0, 0) \in \mathcal{R}_1\} \\
\mathcal{R}_1^\pi &:= \{(f_{R1}, e_{R1}) \mid \exists (f_2, e_2) \; s.t. \; (f_{R1}, e_{R1}, f_2, e_2) \in \mathcal{R}_1\} \\
\mathcal{R}^0 &:= \{(f_{R1}, e_{R1}) \mid (f_{R1}, e_{R1}, 0, 0) \in \mathcal{R}\} \\
\mathcal{R}^\pi &:= \{(f_{R1}, e_{R1}) \mid \exists (f_{R3}, e_{R3}) \; s.t \; ((f_{R1}, e_{R1}, f_{R3}, e_{R3}) \in \mathcal{R})\}.
\end{aligned}$$

Proof. We again follow the same proof as that of achievable Dirac structures, we now define the "copy "$\mathcal{R}_1^*$ of $\mathcal{R}_1$ as

$$\mathcal{R}_1^* := \{(f_{R1}, e_{R1}, f_2, e_2) \mid (-f_{R1}, e_{R1}, -f_2, e_2) \in \mathcal{R}_1\}.$$

Note that $\mathcal{R}_1^*$ is not positive anymore, in fact is a pseudo resistive relation corresponding to negative resistance. Again its clear that $\mathcal{R}_1^*$ is a pseudo resistive relation if and only if $\mathcal{R}_1$ is a resistive relation, and by defining

$$\mathcal{R}_2 := \mathcal{R}_1^* \parallel \mathcal{R}. \tag{4.14}$$

Rest of the proof follows the same procedure as in (4.1) and hence we omit the details here. □

Remark 4.3. It should be noted here that the conditions (4.12) and (4.13) are no longer equivalent as in the case of Dirac structures. This is due to the property of the resistive relation that $\mathcal{R}^\perp = (-\mathcal{R})$, where again $-\mathcal{R}$ is a pseudo resistive relation corresponding to negative resistance.

Remark 4.4. The resistive relation $\mathcal{R}_2$ is obtained by interconnection of a *pseudo* resistive relation $\mathcal{R}_1^*$ with a desired resistive relation $\mathcal{R}$ which is positive. At this point one might think that $\mathcal{R}_2$ might also be a pseudo resistive relation. This need not necessarily be the case. For example consider a case where $\mathcal{R}_1$ is 2 Ohm resistor and the desired $\mathcal{R}$ is of value 1 Ohm. Now this can either be achieved by connecting a pseudo resistance of -1 Ohm in series to $\mathcal{R}_1$, or a 2 Ohm resistance in Parallel. So our conjecture here is that, given a $\mathcal{R}_1$ and a desired $\mathcal{R}$, both satisfying the positivity conditions as in Proposition 3.3, then it is always possible to choose a $\mathcal{R}_2$ which is also positive. That is, there always exists an $\mathcal{R}_2$ which is positive such that Proposition 4.2 is satisfied. One can also consider the following limiting case: Suppose we are given

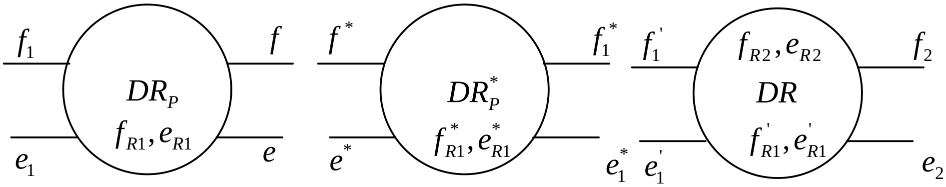


Figure 4.2: $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_P^* \parallel \mathcal{DR}$

a $\mathcal{R}_1$ which is of value $n\ Ohms$, and a desired $\mathcal{R}$ which is $0\ Ohms$.This can be achieved by connecting an "infinite"number of $n\ Ohm$ resistors in parallel with $\mathcal{R}_1$ or with a $0\ Ohm$ resistor in parallel to $\mathcal{R}_1$. On the other hand given a positive $\mathcal{R}_1$ and a desired resistive relation $\mathcal{R}$ which is a pseudo resistive relation, then the $\mathcal{R}_2$ which achieves this is also pseudo in nature.

4.2.3 Achievable Dirac structures with dissipation

We now use the results in the previous two subsections to study the problem of the achievable Dirac structures with dissipation for the closed loop system. We formulate the problem as follows: Given a $\mathcal{D}_p$ with a $\mathcal{R}_p$ (i.e. a plant system with dissipation) and a (to be designed) $\mathcal{D}_c$ with $\mathcal{R}_c$ (a controller system with dissipation) , what are the achievable $(\mathcal{D}_p \parallel \mathcal{R}_p) \parallel (\mathcal{D}_c \parallel \mathcal{R}_c)$. For ease of notation we henceforth use $\mathcal{DR}_p$ for $(\mathcal{D}_p \parallel \mathcal{R}_p)$ and $\mathcal{DR}_c$ for $(\mathcal{D}_c \parallel \mathcal{R}_c)$. Consider here the case where $\mathcal{DR}_p$ is a given Dirac structure with dissipation (finite-dimensional), and $\mathcal{DR}_c$ a to be designed controller Dirac structure with dissipation. We investigate what are the achievable $\mathcal{DR}_p \parallel \mathcal{DR}_c$, the closed-loop structures.

Theorem 4.5. *Given a plant Dirac structure with dissipation $\mathcal{DR}_P$ with port variables $f_1, e_1, f_{R1}, e_{R1}, f, e$ and a desired Dirac structure with dissipation $\mathcal{DR}$ with port-variables $f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}$. Here $(f_1, e_1), (f_{R1}, e_{R1})$ respectively denote the flow and effort variables corresponding to the energy storing elements and the energy dissipating elements of the plant system, similarly with the controller system. Then there exists a controller system $\mathcal{DR}_C$ such that $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$ if and only if the following two conditions are satisfied*

$$\mathcal{DR}_P^0 \subset \mathcal{DR}^0 \tag{4.15}$$

$$\mathcal{DR}^\pi \subset \mathcal{DR}_P^\pi. \tag{4.16}$$

where

$$\begin{aligned}
&\mathcal{DR}_P^0 := \{(f_1, e_1, f_{R1}, e_{R1}) \mid (f_1, e_1, f_{R1}, e_{R1}, 0, 0) \in \mathcal{DR}_P\} \\
&\mathcal{DR}_P^\pi := \{(f_1, e_1, f_{R1}, e_{R1}) \mid \exists (f, e) \ s.t. \ (f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathcal{DR}_P\} \\
&\mathcal{DR}^0 := \{(f_1, e_1, f_{R1}, e_{R1}) \mid (f_1, e_1, f_{R1}, e_{R1}, 0, 0, 0, 0) \in D\mathcal{R}\} \\
&\mathcal{DR}^\pi := \{(f_1, e_1, f_{R1}, e_{R1}) \mid \exists (f_2, e_2, f_{R2}, e_{R2}) \ s.t \\
&\qquad\qquad (f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathcal{DR}\}. \qquad (4.17)
\end{aligned}$$

Proof. The proof is again based on the copy $\mathcal{DR}_P^*$ of the plant system defined as

$$\mathcal{DR}_P^* := \{(f_1, e_1, f_{R1}, e_{R1}, f, e) \mid (-f_1, e_1, -f_{R1}, e_{R1}, -f, e) \in \mathcal{DR}_P\}, \quad (4.18)$$

which is a composition of a Dirac structure and a pseudo-resistive relation. Next define a controller system

$$\mathcal{DR}_C := \mathcal{DR}_P^* \parallel \mathcal{DR}.$$

We follow the same procedure for the proof as in the case of achievable Dirac structures, Proposition 4.1.
Necessity of conditions (4.15) and (4.16) is obvious. Sufficiency is shown by using the controller Dirac structure with dissipation

$$\mathcal{DR}_C := \mathcal{DR}_P^* \parallel \mathcal{DR}.$$

To check that $\mathcal{DR} \subset \mathcal{DR}_P \parallel \mathcal{DR}_C$, consider $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathcal{DR}$. Because $(f_1, e_1, f_{R1}, e_{R1}) \in \mathcal{DR}^\pi$, applying (4.16) yields that $\exists (f, e)$ such that $(f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathcal{DR}_1$. This implies that $(-f_1, e_1, -f_{R1}, e_{R1}, -f, e) \in \mathcal{DR}_P^*$. With the interconnection constraints, see Figure 4.2

$$f = -f^*, \ e = e^*, \ f_1^* = -f_1', \ e_1^* = e_1'.$$

By taking $(f_1', e_1', f_{R_1}', e_{R_1}') = (f_1, e_1, f_{R_1}, e_{R_1})$ in Figure 4.2 it follows that $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C$ and hence $\mathcal{DR} \subset \mathcal{DR}_P \parallel \mathcal{DR}_C$.
To check that $\mathcal{DR}_P \parallel \mathcal{DR}_C \subset \mathcal{DR}$, consider $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C$. Then there exists $f = -f^*$, $e = e^*$, $f_1^* = -f_1'$, $e_1^* = e_1'$ such that

$$\begin{aligned}
&(f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathcal{DR}_1 &(4.19)\\
&(f_1^*, e_1^*, f_{R1}^*, e_{R1}^*, f^*, e^*) \in \mathcal{DR}_1^* \Longleftrightarrow (f_1', e_1', f_{R_1}', e_{R_1}', f, e) \in \mathcal{DR}_P &(4.20)\\
&(f_1', e_1', f_{R1}', e_{R1}', f_2, e_2, e_{R2}, e_{R2}) \in \mathcal{DR}, &(4.21)
\end{aligned}$$

subtracting (4.20) from (4.15) and also by making use of the linearity on $\mathcal{DR}_P$ we get

$$\begin{aligned}
&(f_1 - f_1', e_1 - e_1', f_{R1} - f_{R1}', e_{R1} - e_{R1}', 0, 0) \in \mathcal{DR}_P \Longleftrightarrow \\
&\qquad (f_1 - f_1', e_1 - e_1', f_{R1} - f_{R1}', e_{R1} - e_{R1}') \in \mathcal{DR}_P^0. \qquad (4.22)
\end{aligned}$$

.Using (4.22) and (4.15) we get

$$(f_1 - f_1', e_1 - e_1', f_{R1} - f_{R1}', e_{R1} - e_{R1}', 0, 0, 0, 0) \in \mathcal{DR}. \tag{4.23}$$

Finally, adding (4.23) and (4.21) we get

$$(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathcal{DR},$$

and hence $\mathcal{DR}_P \parallel \mathcal{DR}_C \subset \mathcal{DR}$ □

Remark 4.6. In this case also it can easily be checked that the conditions (4.15) and (4.16) are no more equivalent as in the case of systems without dissipation in Theorem 4.1). This is again due to the compositional property of a Dirac structure with a resistive relation given by $(\mathcal{D} \parallel \mathcal{R})^{\perp} = (\mathcal{D} \parallel -\mathcal{R})$.

Properties of $\mathcal{DR}_P^*$

Consider the following input-state-output port-Hamiltonian plant system with inputs f and outputs e

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\tfrac{\partial H_P}{\partial x}(x) + g(x)f, \quad && x \in \mathcal{X}, f \in \mathbb{R}^m \\ e &= g^T(x)\tfrac{\partial H_P}{\partial x}(x), && e \in \mathbb{R}^m, \end{aligned} \tag{4.24}$$

where $J(x)$ is the interconnection matrix and $R(x)$ corresponds to the dissipation. The corresponding Dirac structure is given by the graph of the map

$$\begin{bmatrix} f_p \\ e \end{bmatrix} = \begin{bmatrix} -[J(x) - R(x)] & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_p \\ f \end{bmatrix}. \tag{4.25}$$

Now, going by the definition of $\mathcal{DR}_P^*$, (see Equation (4.18)) we can write it as

$$\begin{bmatrix} -f_p \\ e \end{bmatrix} = \begin{bmatrix} -[J^*(x) - R^*(x)] & -g^*(x) \\ g^{*T}(x) & 0 \end{bmatrix} \begin{bmatrix} e_p \\ -f \end{bmatrix}. \tag{4.26}$$

This implies that the interconnection matrix $J^*(x)$, the dissipation matrix $R^*(x)$ and the input vector field $g^*(x)$ of $\mathcal{DR}_P^*$ would relate to the interconnection matrix $J(x)$, the dissipation matrix $R(x)$ and the input vector field $g(x)$ of $\mathcal{DR}_P$ as follows

$$\begin{aligned} &J^*(x) = -J(x) \quad R^*(x) = -R(x) \\ &g^*(x) = g(x). \end{aligned} \tag{4.27}$$

A standard plant-controller interconnection would result in a closed-loop Dirac structure of the form, which we call as the desired closed-loop system

$$\begin{aligned} \begin{bmatrix} f_p \\ f_c \end{bmatrix} &= \begin{bmatrix} -J(x) & g(x)g_c^T(\xi) \\ -g_c(\xi)g^T(x) & -J_c(\xi) \end{bmatrix} + \begin{bmatrix} R(x) & 0 \\ 0 & R_c(\xi) \end{bmatrix} \begin{bmatrix} e_p \\ e_c \end{bmatrix} \\ \begin{bmatrix} e \\ \tilde{e} \end{bmatrix} &= \begin{bmatrix} g^T(x) & 0 \\ 0 & g_c^T(\xi) \end{bmatrix} \begin{bmatrix} e_p \\ e_c \end{bmatrix}. \end{aligned} \tag{4.28}$$

It can easily be checked that such a Dirac structure would satisfy the conditions (4.15,4.16) and hence we can construct a controller Dirac structure as in Theorem 4.5. The controller Dirac structure is defined as $\mathcal{DR}_C = \mathcal{DR}_P^* \parallel \mathcal{DR}$. Interconnecting $\mathcal{DR}_P^*$ and $\mathcal{DR}$ with the following interconnection constraints

$$f_p^* = -f_p$$
$$e_p^* = e_p,$$

would result in the following

$$\begin{aligned} f_c &= -[J_c(\xi) - R_c(\xi)]e_c - g_c(\xi)g^T(x)e_p \\ g(x)g_c^T(\xi)e_c &= -g(x)f. \end{aligned} \tag{4.29}$$

We know from (4.28) that

$$e = g^T(x)e_p = \tilde{f}.$$

Also , due to the left invertibility of $g(x)$, we have the following

$$g_c^T(\xi)e_c = f = \tilde{e},$$

and hence we can rewrite (4.29) as

$$\begin{aligned} f_c &= -[J_c(\xi) - R_c(\xi)]e_c - g_c(\xi)\tilde{f} \\ \tilde{e} &= -f, \end{aligned} \tag{4.30}$$

which gives the controller Dirac structure, with the input of the controller given by the output of the plant system and the output of the controller given by negative of the plant input, the case of such interconnection is called the gyrative interconnection. It then directly follows that $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$.

4.2.4 Casimirs for a system with dissipation

We define a Casimir for a port-Hamiltonian system with dissipation to be any function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ such that its gradient, $e = \frac{\partial \mathcal{C}}{\partial x}$, satisfies

$$(0, e, 0, 0) \in \mathcal{DR},$$

which implies that

$$\frac{d\mathcal{C}}{dt} = e^T f_p = 0. \tag{4.31}$$

At this point one may think that the definition of Casimir function may be relaxed by requiring that the above expression holds only for a specific resistive relation

$$R_f f_R + R_e e_R = 0, \tag{4.32}$$

where the square matrices R_f and R_e satisfy the symmetry and positive semi definiteness condition

$$R_f R_e^T = R_e R_f^T \geq 0,$$

together with the dimensionality condition

$$\text{rank}[R_f \mid R_e] = \dim f_R.$$

In this case, the condition for a function to be a conserved quantity for one resistive relation will actually imply that it is a conserved quantity for all resistive relations.

Indeed, let $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ be a function satisfying (4.31) for a specific resistive port $\mathcal{R}$ specified by matrices R_f and R_e as above. This means that $e = \frac{\partial \mathcal{C}}{\partial x}(x)$ satisfies

$$\begin{aligned} &e^T f_p = 0, \forall f_p \text{ for which } \exists e_p, f_R, e_R \; s.t \; (f_p, e_p, f_R, e_R) \in \mathcal{DR} \\ &\text{and } R_f f_R + R_e e_R = 0. \end{aligned} \tag{4.33}$$

However, this implies that $(0, e, 0, 0) \in (\mathcal{D} \parallel \mathcal{R})^\perp$. We also know that $(\mathcal{D} \parallel \mathcal{R})^\perp = \mathcal{D} \parallel (-\mathcal{R})$, and thus there exists $\tilde{f}_R, \tilde{e}_R$ such that $R_f \tilde{f}_R - R_e \tilde{e}_R = 0$ and

$$(0, e, \tilde{f}_R, \tilde{e}_R) \in \mathcal{DR}.$$

Hence,

$$0 = e^T \cdot 0 + \tilde{e}_R^T \tilde{f}_R = \tilde{e}_R^T \tilde{f}_R.$$

By writing the pseudo resistive relation $-\mathcal{R}$ in image representation [58], $\tilde{f}_R = R_e^T \lambda$, $\tilde{e}_R = R_f^T \lambda$, it follows that

$$\lambda^T R_f R_e^T \lambda = 0,$$

and by the positive definiteness condition $R_f R_e^T = R_e R_f^T > 0$ this implies that $\lambda = 0$, whence $\tilde{f}_R = \tilde{e}_R = 0$. Hence not only $(0, e, \tilde{f}_R, \tilde{e}_R) \in \mathcal{D}$ but actually $(0, e, 0, 0) \in \mathcal{D}$, implying that e is the gradient of the Casimir function.

Of course, the above argument does not fully carry through if the resistive relations are only positive semi-definite. In particular this is the case if $R_f R_e^T = 0$ (implying zero dissipation), corresponding to the presence of ideal power-conserving constraints.

4.2.5 Achievable Casimirs for any resistive relation

In this section we characterize the achievable Casimirs of the closed-loop system, in terms of the plant state. This characterization in terms of plant state is useful in the sense that given a plant Dirac structure we can, without defining a controller, determine whether or not there exist Casimir functions for the

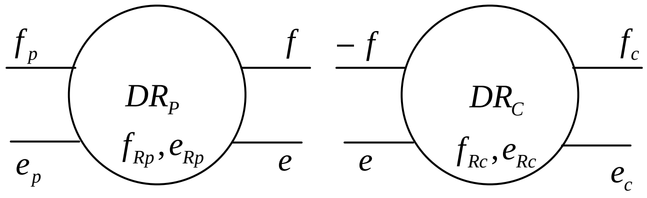


Figure 4.3: $\mathcal{DR}_P \parallel \mathcal{DR}_C$

closed-loop system which will shown will the help of some examples later on. This is in addition to the fact the we can also determine the Casimir functions for all R and R_c, with $R \geq 0$ and $R_c \geq 0$

We now consider the question of characterizing the set of achievable Casimirs for the closed-loop system $\mathcal{DR}_P \parallel \mathcal{DR}_C$ for all resistive relations and every port behavior. Here $\mathcal{DR}_P$ is the Dirac structure of the plant port-Hamiltonian system with dissipation with Hamiltonian H_P, and $\mathcal{DR}_C$ is the controller Dirac structure. Then the Casimirs depend on the plant state x and also on the controller state ξ, with the controller Hamiltonian $H_C(\xi)$ at our own disposal.

Consider the notation as in Figure 4.3 and assume that the ports in (f_p, e_p), (f_{R_p}, e_{R_p}) are respectively connected to the (given) energy storing elements and the energy dissipating elements of the plant port-Hamiltonian system. Similarly (f_c, e_c) are connected to the (to be designed) energy storing elements of the controller port-Hamiltonian system with dissipation; that is $(f_c = -\dot{\xi}, e_c = \frac{\partial^T H_C}{\partial \xi})$ and (f_{R_c}, e_{R_c}) are connected to the energy dissipation elements of the controller system. In this situation the achievable Casimirs are functions $\mathcal{C}(x, \xi)$ such that $e_p = \frac{\partial^T \mathcal{C}}{\partial x}(x, \xi)$, $e_c = \frac{\partial^T \mathcal{C}}{\partial \xi}(x, \xi)$ belongs to the space

$$P_{Cas} = \{e_p \mid \exists \mathcal{DR}_c \; s.t \; \exists e_c : (0, e_p, 0, 0, 0, e_c, 0, 0) \in \mathcal{DR}_P \parallel \mathcal{DR}_C\}. \tag{4.34}$$

The following theorem addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x by characterization of the space P_{Cas}.

Proposition 4.7. *The space P_{Cas} defined above is equal to the space*

$$\tilde{P} = \{e_p \mid \exists (f, e) \; s.t \; (0, e_p, 0, 0, f, e) \in \mathcal{DR}_P\}.$$

Proof. We see that $P_{Cas} \subset \tilde{P}$ trivially and by using the controller Dirac structure $\mathcal{DR}_C = \mathcal{DR}_P^*$ we obtain $\tilde{P} \subset P_{Cas}$. $\square$

4.2.6 Achievable Casimirs for a given resistive relation

If $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ is a Casimir function for a specific resistive relation $\mathcal{R}$ given by (4.32), then this means that $e = \frac{\partial \mathcal{C}}{\partial x}(x)$ satisfies

$$\frac{\partial^T \mathcal{C}}{\partial x}(x) f_p = 0, \text{ for all } f_p \text{ s.t } \exists e_s, f_R, e_R \text{ s.t } (f_p, e_p, f_R, e_R) \in \mathcal{DR}$$
$$\text{and } R_f f_R + R_e e_R = 0,$$

which means that $(0, e, 0, 0) \in (\mathcal{D} \parallel \mathcal{R})^{\perp}$ (refer Equation (4.33)). Since we know by Proposition 3.3 that $(\mathcal{D} \parallel \mathcal{R})^{\perp} = \mathcal{D} \parallel -\mathcal{R}$, and thus C is a Casimir if there exist (f_R, e_R) such that

$$(0, e, -f_R, e_R) \in \mathcal{DR}.$$

We now consider the question of finding all the achievable Casimirs for the closed-loop system $\mathcal{DR}_P \parallel \mathcal{DR}_C$, with $\mathcal{DR}_P$ the Dirac structure of the plant port-Hamiltonian system with dissipation with Hamiltonian H_P, and $\mathcal{DR}_C$ is the controller Dirac structure; for a given resistive relations and every port behavior. Consider $\mathcal{DR}_P$ and $\mathcal{DR}_C$ as above, and in this case the achievable Casimirs are functions $\mathcal{C}(x, \xi)$ such that $\frac{\partial^T \mathcal{C}}{\partial x}(x, \xi)$ belongs to the space

$$\begin{aligned} P_{Cas} = \quad & \{e_p \mid \exists \mathcal{DR}_c \; s.t \; \exists e_c, f_{Rp}, e_{Rp}, f_{Rc}, e_{Rc} : \\ & (0, e_p, -f_R, e_R, 0, e_c, -f_{Rc}, e_{Rc}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C\}. \end{aligned}$$

Proposition 4.8. *The space P_{Cas} defined above is equal to the linear space*

$$\tilde{P} = \{e_1 \mid \exists (f_{Rp}, e_{Rp}, f, e) \; s.t \; (0, e_p, -f_{Rp}, e_{Rp}, f, e) \in \mathcal{DR}_P\}.$$

Proof. The proof follows the same procedure as in Proposition 4.7. □

Example 4.9. Consider the port-Hamiltonian system with (f_p, e_p) respectively the flows and efforts corresponding to the energy storage elements, (f_R, e_R) the flows and efforts corresponding to the energy dissipating elements and inputs f and outputs e. The corresponding Dirac structure is given by

$$\begin{aligned} f_p &= -J(x)e_p - g_R(x) f_R - g(x) f \\ \begin{bmatrix} e_R \\ e \end{bmatrix} &= \begin{bmatrix} g_R^T(x) \\ g^T(x) \end{bmatrix} e_p. \end{aligned}$$

The characterization of the space P_{Cas} is given by

$$\begin{aligned} P_{Cas} = \{e_p \mid \exists f_p \text{ s.t } 0 = -J(x)e_p - g(x) f \text{ and} \\ 0 = g_R^T(x) e_p\}. \end{aligned} \tag{4.35}$$

The above expression implies that the achievable Casimirs do not depend on the coordinate where dissipation enters into the system (follows from the second line, and well known in literature as the "dissipation obstacle"). In addition to that they are also the Hamiltonian functions corresponding to the input vector fields given by the columns of $g(x)$.

Example 4.10 (The Series RLC circuit). The dynamics of the circuit are given by

$$\begin{bmatrix} \dot{q} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} \frac{q}{C} \\ \frac{\phi}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

and the corresponding Dirac structure is given by

$$\begin{aligned} \begin{bmatrix} -\dot{q} \\ -\dot{\phi} \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{q}{C} \\ \frac{\phi}{L} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ R\frac{\phi}{L} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ e_R &= \frac{\phi}{L} \\ e_p &= \begin{bmatrix} 0 & \frac{x\phi}{L} \end{bmatrix}^T . \end{aligned}$$

Comparing with Example 4.9 we have

$$(f_p, e_p, f_R, e_R, f, e) = \left(-[\dot{q}, \dot{\phi}]^T, [\tfrac{q}{C}, \tfrac{\phi}{L}]^T, [0 \text{ -} R\tfrac{\phi}{L}]^T, [0 \ \tfrac{\phi}{L}]^T, u, e \right);$$

$$g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad g_R(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

In this case the achievable Casimirs (in terms of the plant state $x = [q, \phi]^T$) should satisfy the following set of equations

$$\frac{\partial C}{\partial \phi}(x, \xi) = 0.$$

The above expression is a constraint on the Casimir function implying that any Casimir function for this system does not depend on ϕ term, which is precisely where dissipation enters into the system. There however, would exist a Casimir which depends on the q term.

Example 4.11 (The Parallel RLC circuit). We next consider the case of a parallel RLC circuit whose dynamics are given by the following set of equations

$$\begin{bmatrix} \dot{q} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{q}{C} \\ \frac{\phi}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

and the corresponding Dirac structure given by

$$\begin{bmatrix} -\dot{q} \\ -\dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{q}{C} \\ \frac{\phi}{L} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{q}{RC} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

where

$$(f_p, e_p, f_R, e_R, f, e) = \Big(-[\dot{q}, \dot{\phi}]^T, [\tfrac{q}{C}, \tfrac{\phi}{L}]^T, [\tfrac{q}{RC}\ 0]^T, [0\ \tfrac{\phi}{L}]^T, u, e\Big);$$

$$g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad g_R(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

As above the achievable Casimirs in terms of the plant state x should be such that

$$\frac{\partial C}{\partial q}(x, \xi) = \frac{\partial C}{\partial \phi}(x, \xi) = 0,$$

which means that we cannot find any Casimir functions for the closed-loop system which depend on the plant state x (the only possible Casimirs are the ″trivial Casimirs″which are constant).

Example 4.12 (Special Case of Example 2.9). Consider the Capacitor microphone as in Example 2.9, now with $F = 0$. (Such a case as considered as an example in [36].) The elements of the Dirac structure in case of the Capacitor microphone would be as follows

$$\begin{aligned} f_p &= -[\dot{q}\ \dot{p}\ \dot{Q}]^T, e_p = [\frac{\partial H}{\partial q}\ \frac{\partial H}{\partial p}\ \frac{\partial H}{\partial Q}] \\ f_R &= [0\ \ -c\frac{\partial H}{\partial p}\ \text{-}\frac{1}{R}\frac{\partial H}{\partial Q}]^T, e_R = [0\ \frac{\partial H}{\partial p}\ \frac{\partial H}{\partial Q}]^T \\ g(x) &= \begin{bmatrix} 0 \\ 0 \\ 1/R \end{bmatrix};\ g_R(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The achievable Casimirs in terms of the plant state are all functions $C(q, p, Q)$, satisfying

$$\begin{aligned} \frac{\partial C}{\partial q} &= 0 \\ \frac{\partial C}{\partial p} &= 0 \\ \frac{\partial C}{\partial Q} &= 0. \end{aligned}$$

This means that we cannot find Casimirs depending on the plant state.

Remark 4.13. The existence of Casimirs is not a guarantee that the closed-loop system has the desired stability properties. This is typically the case for example in the case of underactuated mechanical systems and a broad class of electromechanical systems where the Casimirs, even though they exist, are not functions of the coordinates that need to be "shaped". We shall elaborate on this in the next chapter on control of port-Hamiltonian systems and also discuss a few possible ways to overcome this drawback.

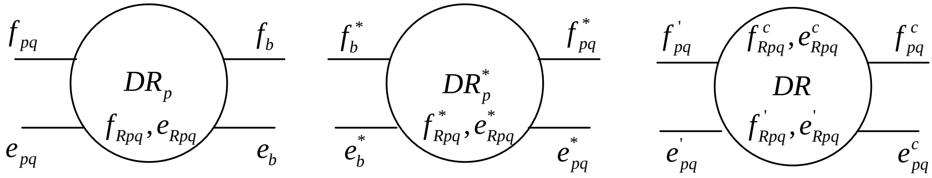


Figure 4.4: $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_P^* \parallel \mathcal{DR}$

4.3 Achievable Casimirs for infinite-dimensional systems

4.3.1 Achievable Dirac structures

Similar to the finite-dimensional case we investigate what are the achievable closed-loop Dirac structures interconnecting a given plant Stokes-Dirac structure with dissipation $\mathcal{DR}_P$ to a to be designed controller Stokes-Dirac structure with dissipation $\mathcal{DR}_C$.

Theorem 4.14. *Given a plant Stokes-Dirac structure with dissipation $\mathcal{DR}_P$, with port variables $(f_{pq}, f_{R_{pq}}, f_b, e_{pq}, e_{R_{pq}}, e_b)$ and a desired $\mathcal{DR}$ with port variables $(f_{pq}, f_{R_{pq}}, f_{pq}^c, f_{R_{pq}}^c, e_{pq}, e_{R_{pq}}, e_{pq}^c, e_{R_{pq}}^c)$. A certain interconnected $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}$ can be achieved by a proper choice of the controller Stokes-Dirac structure with dissipation if and only if the following two conditions are satisfied*

$$\mathcal{DR}_P^0 \subset \mathcal{DR}^0 \tag{4.36}$$

$$\mathcal{DR}^\pi \subset \mathcal{DR}_P^\pi, \tag{4.37}$$

where

$$\begin{aligned}
\mathcal{DR}_P^0 &:= \{(f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, 0, 0) \in \mathcal{DR}_p\} \\
\mathcal{DR}_P^\pi &:= \{(f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \exists (f_b, e_b) : (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_b, e_b) \in \mathcal{DR}_P\} \\
\mathcal{DR}^0 &:= \{(f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, 0, 0, 0, 0) \in \mathcal{DR}\} \\
\mathcal{DR}^\pi &:= \{(f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \exists (f_{pq}^c, e_{pq}^c, f_{R_{pq}}^c, e_{R_{pq}}^c) \\
&\qquad\qquad : (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_{pq}^c, e_{pq}^c, f_{R_{pq}}^c, e_{R_{pq}}^c) \in \mathcal{DR}\}.
\end{aligned}$$

Proof. The proof follows the same lines as in the finite-dimensional case, which again is based on a "copy"of $\mathcal{D}_p$ (also see Figure 4.4) which in this case is defined as

$$\mathcal{DR}_P^* := \{(f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_b, e_b) \mid (-f_{pq}, e_{pq}, -f_{R_{pq}}, e_{R_{pq}}, -f_b, e_b) \in \mathcal{DR}_P\}. \tag{4.38}$$

□

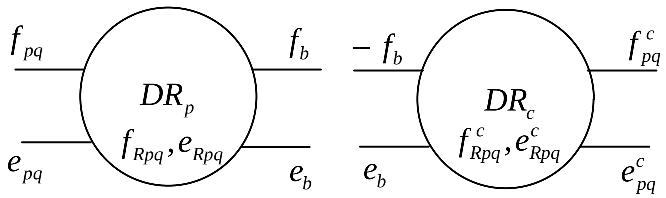


Figure 4.5: $\mathcal{DR}_P \parallel \mathcal{DR}_C$

Properties of $\mathcal{DR}_P^*$

Consider an infinite-dimensional port-Hamiltonian system with a 1-D spatial domain and a distributed dissipation defined with respect to a Stokes-Dirac structure $\mathcal{DR}_P$ given by

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} G* & \mathrm{d} \\ \mathrm{d} & R* \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_p \mid \partial Z \\ e_q \mid \partial Z \end{bmatrix}. \tag{4.39}$$

Now, consider the following closed-loop (achievable) Dirac structure $\mathcal{DR}$. This is obtained by interconnecting this system to another infinite-dimensional port-Hamiltonian system

$$\begin{bmatrix} f_p \\ f_q \\ f_p^c \\ f_q^c \end{bmatrix} = \begin{bmatrix} G* & \mathrm{d} & 0 & 0 \\ \mathrm{d} & R* & 0 & 0 \\ 0 & 0 & G^c* & \mathrm{d} \\ 0 & 0 & \mathrm{d} & R^c* \end{bmatrix} \begin{bmatrix} e_p \\ e_q \\ e_p^c \\ e_q^c \end{bmatrix}. \tag{4.40}$$

It can easily be checked that this Dirac structure satisfies the conditions (4.36,4.37). By the definition of $\mathcal{DR}_P^*$ from Equation (4.38), we can write it as

$$\begin{bmatrix} -f_p \\ -f_q \end{bmatrix} = \begin{bmatrix} -G* & -\mathrm{d} \\ -\mathrm{d} & -R* \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}$$
$$\begin{bmatrix} -f_b \\ -e_b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_p \mid \partial Z \\ e_q \mid \partial Z \end{bmatrix}. \tag{4.41}$$

Theorem 4.14 defines the controller Dirac structure with dissipation as $\mathcal{DR}_C = \mathcal{DR}_P^* \parallel \mathcal{DR}$. Now, interconnecting $\mathcal{DR}_P$ with $\mathcal{DR}$ with the following interconnection constraints

$$\begin{array}{lll} f_p^* = -f_p, & f_q^* = -f_q, & f_b^* = -f_b, \\ e_p^* = e_p, & e_q^* = e_q, & e_b^* = e_b, \end{array} \tag{4.42}$$

would, with the help of a few computations, result in the following controller Stokes-Dirac structure with dissipation

$$\begin{bmatrix} f_p^c \\ f_q^c \end{bmatrix} = \begin{bmatrix} G^c* & \mathrm{d} \\ \mathrm{d} & R^c* \end{bmatrix} \begin{bmatrix} e_p^c \\ e_q^c \end{bmatrix}$$
$$\begin{bmatrix} f_b^c \\ e_b^c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_p^c \mid_{\partial Z} \\ e_q^c \mid_{\partial Z} \end{bmatrix}. \tag{4.43}$$

It then immediately follows that $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$.

4.3.2 Casimirs for an infinite-dimensional system

Consider the distributed parameter port-Hamiltonian system without dissipation on an $n-$dimensional spatial domain Z having state space $\Omega^p(Z) \times \Omega^q(Z)$ and Stokes-Dirac structure given by (2.46). The Casimirs for this system, which are independent of the Hamiltonian are obtained as follows. Let

$$\mathcal{C} : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \mathbb{R},$$

be a function satisfying

$$\mathrm{d}(\delta_p \mathcal{C}) = 0, \;\; \mathrm{d}(\delta_q \mathcal{C}) = 0. \tag{4.44}$$

Then the time-derivative of C along the trajectories of the system is given as

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= \int_Z [\delta_p \mathcal{C} \wedge \dot{\alpha}_p + \delta_q \mathcal{C} \wedge \dot{\alpha}_q] \\ &= \int_Z \delta_p \mathcal{C} \wedge (-1)^r \mathrm{d}(\delta_q H) - \int_Z \delta_q \mathcal{C} \wedge (-1)^r \mathrm{d}(\delta_p H) \\ &= -(-1)^{n-q} \int_Z \mathrm{d}(\delta_q H \wedge \delta_p \mathcal{C}) - (-1)^{n-q} \int \mathrm{d}(\delta_q \mathcal{C} \wedge \delta_p H) \\ &= \int_{\partial Z} [e_b \wedge f_b^C + e_b^C \wedge f_b], \end{aligned}$$

where we have denoted

$$f_b^C := \delta_p \mathcal{C} \mid_{\partial Z}, \quad e_b^C := -(-1)^{n-q} \delta_q \mathcal{C} \mid_{\partial Z} . \tag{4.45}$$

In particular, if in addition to (4.44) the function C satisfies

$$\delta_p \mathcal{C} \mid_{\partial Z} = 0, \quad \delta_q \mathcal{C} \mid_{\partial Z} = 0,$$

then $\frac{d\mathcal{C}}{dt} = 0$ along the system trajectories for any Hamiltonian H. The function $\mathcal{C}$ satisfying (4.44) and (4.45) is called a Casimir function for the infinite-dimensional port-Hamiltonian system (2.52). If C satisfies Equation (4.44) but not Equation (4.45) then $\mathcal{C}$ is called a conservation law for the system, as its time-derivative is determined by the boundary conditions of the system.

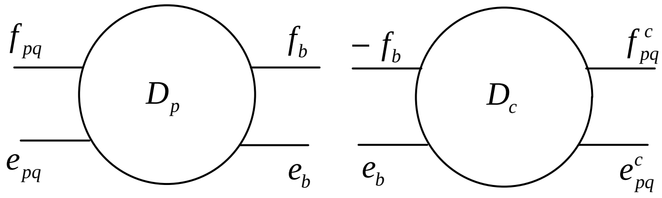


Figure 4.6: $\mathcal{D}_P \parallel \mathcal{D}_C$

4.3.3 Achievable Casimirs for systems without dissipation

In the previous subsection we have defined a Casimir function for an infinite-dimensional system. If we write the Equation (4.44) in terms of the elements of the Stokes-Dirac structure, we can say that a Casimir function for an *autonomous* infinite-dimensional port-Hamiltonian system with dissipation is any functional $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ such its gradients $e_p = \delta_p \mathcal{C}$, $e_q = \delta_q \mathcal{C}$ satisfy

$$(0, 0, e_p, e_q) \in \mathcal{D},$$

implying that

$$\frac{d\mathcal{C}}{dt} = \int_Z e_p \wedge f_p + e_q \wedge f_q = 0.$$

We now address the question of characterizing the set of achievable Casimirs for the closed-loop system. Consider the notation in Figure 4.6, where $\mathcal{DR}_P$ and $\mathcal{DR}_C$ are two infinite-dimensional port-Hamiltonian systems, interconnected to each other through the boundary. The interconnection can either be through a part or whole of the boundary. Assume ports in (f_{pq}, e_{pq}) are connected to the (given) energy storing elements of the plant port-Hamiltonian system and (f_b, e_b) the boundary port variables. Similarly (f^c_{pq}, e^c_{pq}) connected to the to-be-designed energy storing elements of the controller port-Hamiltonian system with (f^c_b, e^c_b) being the boundary variables of the controller system. In this situation the achievable Casimir functions are functions $\mathcal{C}$ such that $e_p = \delta_p \mathcal{C}$, $e_q = \delta_q \mathcal{C}$ belongs to the space

$$P_{Cas} = \left\{ e_{pq} \mid \exists \mathcal{D}_C \ s.t \ \exists \ e^c_{pq} : (0, e_{pq}, 0, e^c_{pq}) \in \mathcal{D}_P \| \ \mathcal{D}_C \right\}. \tag{4.46}$$

Again for brevity we use e_{pq} for (e_p, e_q). The following theorem then addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x , by finding a characterization of the space P_{Cas}

Proposition 4.15. *The space P_{Cas} defined in (4.46) is equal to the linear space*

$$\tilde{P} = \{ e_{pq} \mid \exists (f_b, e_b) : (0, e_{pq}, f_b, e_b) \in \mathcal{D}_P \}.$$

Proof. The inclusion $P_{Cas} \subset \tilde{P}$ is obvious, and taking the controller Dirac structure $\mathcal{D}_c = \mathcal{D}_p^*$, the second inclusion $\tilde{P} \subset P_{Cas}$ is obtained.
Since for all $e_{pq} \in \tilde{P}$ we have $f_{pq} = 0$ and $(0, e_{pq}, -f_b, e_b) \in \mathcal{D}_p^*$. And with respect to the Stokes-Dirac structure (2.46) this would mean that the space $\tilde{P}$ is such that e_p and e_q are constants as functions of the spatial variable, which in addition would mean $f_0 = f_l$ and $e_0 = e_l$, thus resulting in finite-dimensional controllers. □

Example 4.16. We see that in case of the transmission line without dissipation (2.55), the Casimirs are functionals $\mathcal{C}$ such that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{C} \\ \delta_\phi \mathcal{C} \end{bmatrix}, \tag{4.47}$$

which means that the set of achievable Casimir functions is such that $\delta_q \mathcal{C}$ and $\delta_\phi \mathcal{C}$ are constant as a function of z which means that

$$\delta_q \mathcal{C} \mid_0 = \delta_q \mathcal{C}(z) = \delta_q \mathcal{C} \mid_l, \quad \delta_\phi \mathcal{C} \mid_0 = \delta_\phi \mathcal{C}(z) = \delta_\phi \mathcal{C} \mid_l \ ,$$

or in other words every Casimir function should be linear with respect to the spatial variables.

Example 4.17. Consider the case of the shallow water equations with the additional velocity component (2.69) $\mathcal{C}$ is a Casimir function if its gradients satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} & 0 \\ \mathrm{d} & 0 & \frac{1}{*h}\mathrm{d}(*v) \\ 0 & -\frac{1}{*h}\mathrm{d}(*v) & 0 \end{bmatrix} \begin{bmatrix} \delta_h \mathcal{C} \\ \delta_u \mathcal{C} \\ \delta_v \mathcal{C} \end{bmatrix}.$$

It follows from the first and the third rows of the above matrix that

$$\delta_u \mathcal{C} = 0,$$

meaning that the Casimir functions do not depend on the u component of the velocity. To find all the Casimirs of the system we need to solve the PDE given by the second row of the matrix i.e.

$$\mathrm{d}\delta_h \mathcal{C} = \frac{1}{h}\mathrm{d}(*v)\delta_v \mathcal{C}. \tag{4.48}$$

It can be shown that [53] that all the functions of the form given below are Casimirs for the system

$$\mathcal{C} = \int_Z h \cdot (\phi(\frac{1}{*h}\mathrm{d}(*v))). \tag{4.49}$$

Following are a few examples of Casimir functions:

1. Case where $\phi(\frac{1}{*h}\mathrm{d}(*v)) = 1$, in which case have $\mathcal{C} = \int_Z h$ which corresponds to mass conservation
2. For $\phi(\frac{1}{*h}\mathrm{d}(*v)) = \frac{1}{*h}\mathrm{d}(*v)$, we have $\mathcal{C} = \int_Z \mathrm{d}(*)v$ which corresponds to *vorticity*.
3. The case where $\phi(\frac{1}{*h}\mathrm{d}(*v)) = (\frac{1}{*h}\mathrm{d}(*v))^2$, we have $\mathcal{C} = \int_Z \frac{1}{*h} * (\mathrm{d}(*v)) \wedge (\mathrm{d}(*v))$ which is called the *mass weighted potential enstrophy*.

4.3.4 Achievable Casimirs for systems with dissipation

In this section we consider the case where now we have dissipation into the infinite-dimensional system. From the finite-dimensional analysis we know that a Casimir function for a given resistive relation is such that its gradients $e = \frac{\partial \mathcal{C}}{\partial x}(x)$satisfy (4.33) which implies that

$$(0, e) \in (\mathcal{D} \parallel \mathcal{R})^{\perp} = (\mathcal{D} \parallel -\mathcal{R}).$$

Analogously in the case of infinite-dimensional systems with dissipation we define a Casimir to be a functional such that its gradients ($e = \delta\mathcal{C}$) satisfy

$$\begin{array}{l}\int e_p \wedge f_p + e_q \wedge f_q = 0 \; \forall (f_{pq}) \text{ for which} \\ \exists e_{pq}, f_{R_{pq}}, e_{R_{pq}} \; s.t.(f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \in \mathcal{DR},\end{array}$$

and the resistive relation satisfying (3.9). This means that $(0, e_{pq}) \in (\mathcal{D} \parallel \mathcal{R})^{\perp} = (\mathcal{D} \parallel -\mathcal{R})$. Thus we can say that a functional is a Casimir if its gradients $e_{pq} = \delta_{pq}\mathcal{C}$ satisfy

$$(0, e_{pq}, -f_{R_{pq}}, e_{R_{pq}}) \in \mathcal{DR}.$$

To address the question of finding all the achievable Casimirs for the closed loop system $\mathcal{DR}_P \parallel \mathcal{DR}_C$, we consider the case where both $\mathcal{DR}_P$ and $\mathcal{DR}_C$ are infinite-dimensional port-Hamiltonian systems with dissipation, and are defined with respect to a Stokes-Dirac structure. The interconnection between $\mathcal{DR}_P$ and $\mathcal{DR}_C$ takes place through the boundary of the system (see Figure 4.5). It can easily be shown that such an interconnection (through the boundary) leads to another port-Hamiltonian system with dissipation. In this case the achievable Casimirs are functionals $\mathcal{C}$ such that $e_{pq} = \delta_{pq}\mathcal{C}$ belongs to the space

$$P_{Cas} = \{e_{pq} \mid \exists \mathcal{DR}_C \; s.t \; \exists e^c_{pq} : (0, e_{pq}, -f_{R_{pq}}, e_{R_{pq}}, 0, e^c_{pq} - f^c_{R_{pq}}, e^c_{R_{pq}} \in \mathcal{D}_P \parallel \mathcal{D}_C\}.$$

The characterization of the set of achievable Casimirs of the closed-loop system in terms of the plant state, by finding characterization of the space P_{Cas}, is addressed by the following theorem

Proposition 4.18. *The space P_{Cas} defined above is equal to the linear space*

$$\tilde{P} = \{e_{pq} \mid \exists (f_b, e_b) : (0, e_{pq}, -f_{Rpq}, e_{Rpq}, f_b, e_b) \in \mathcal{D}_P\}.$$

Proof. The proof follows the same steps as before by taking $\mathcal{DR}_C = \mathcal{DR}_P^*$,where $\mathcal{DR}_P^*$ is as defined above. □

Example 4.19. The dynamics of the transmission line with dissipation are given by

$$\begin{bmatrix} -\partial_t q(z,t) \\ -\partial_t \phi(z,t) \end{bmatrix} = \left(\begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} + \begin{bmatrix} G* & 0 \\ 0 & R* \end{bmatrix} \right) \begin{bmatrix} \delta_q H \\ \delta_\phi H \end{bmatrix}$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_q H \mid_{\partial Z} \\ \delta_\phi H \mid_{\partial Z} \end{bmatrix}.$$

Here G and R respectively represent the distributed conductance and distributed resistance in the transmission line see Equation (3.15). Now, by applying Proposition 4.18, we see that the achievable Casimirs are all functionals $\mathcal{C}(q(z,t), \phi(z,t))$ which satisfy

$$\begin{aligned} \mathrm{d}\delta_\phi C - G * \delta_q C &= 0 \\ \mathrm{d}\delta_q C - R * \delta_\phi C &= 0. \end{aligned} \tag{4.50}$$

Remark 4.20. Contrast to the case of a transmission line without dissipation (4.47), the clear distinction here is that we do not have Casimirs which are constant with respect to the spatial variable z. This is clearly due to the presence of dissipation in the transmission line.

4.4 Achievable Casimirs for mixed finite and infinite-dimensional systems

4.4.1 Achievable Dirac structures

The mixed finite and infinite-dimensional case we will consider here (and the rest of the section) is the case where the plant Dirac structure $\mathcal{DR}_P$ is the interconnection of a Stokes-Dirac structure with a finite-dimensional Dirac structure connected to one of its boundary, the controller Dirac structure $\mathcal{DR}_C$ being a finite-dimensional Dirac structure connected to the other end of the Stokes-Dirac structure. In terms of Figure 3.5 this would mean a case where $\mathcal{D}_P = \mathcal{D}_1 \parallel \mathcal{D}_\infty$ together with their respective resistive relations and $\mathcal{D}_C = \mathcal{D}_2$ also with its resistive relation. This typically is a case where we wish to control a plant which is interconnected to a controller through an infinite-dimensional system. An example is a power-drive consisting of a power converter, transmission line and electrical machine.

Finding all the achievable Dirac structures of the closed-loop system in this case follows, by a combination of the procedures followed in the previous subsections. We briefly highlight the problem here.

Corollary 4.21. *Consider a given mixed plant port-Hamiltonian system with dissipation $\mathcal{DR}_P$ with port-variables $(f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}})$, which is a composition of a Stokes-Dirac structure with a finite-dimensional Dirac structure connected to one end of its boundary. Here (f_1, e_1), (f_{R1}, e_{R1}) respectively denote the flow and effort variables corresponding to the energy storing elements and the energy dissipating elements of the finite-dimensional part of the plant subsystem. Similarly (f_{pq}, e_{pq}), $(f_{R_{pq}}, e_{R_{pq}})$ respectively denote the flow and effort variables corresponding to the energy storing elements and the energy dissipating elements of the infinite-dimensional part of the plant subsystem. In this case a certain interconnected $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$, where $\mathcal{DR}_C$ is a to-be-designed finite-dimensional controller Dirac structure, can be achieved by proper choice of the controller Dirac structure if and only if the following two conditions are satisfied*

$$\mathcal{DR}_P^0 \subset \mathcal{DR}^0 \tag{4.51}$$
$$\mathcal{DR}^\pi \subset \mathcal{DR}_P^\pi, \tag{4.52}$$

where

$$\begin{aligned}
\mathcal{DR}_P^0 := \quad & \{(f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \\
& (f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, 0, 0) \in \mathcal{DR}_p\} \\
\mathcal{DR}_P^\pi := \quad & \{(f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \\
& \exists (f_l, e_l) : (f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_l, e_l) \in \mathcal{DR}_P\} \\
\mathcal{DR}^0 := \quad & \{(f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \\
& (f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, 0, 0, 0, 0) \in \mathcal{DR}\} \\
\mathcal{DR}^\pi := \quad & \{(f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \exists (f_2, e_2, f_{R_2}, e_{R_2}) : \\
& (f_1, e_1, f_{R_1}, e_{R1}, f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_2, e_2, f_{R_2}, e_{R_2}) \in \mathcal{DR}\}.
\end{aligned} \tag{4.53}$$

4.4.2 Achievable Casimirs

In this case the achievable Casimirs are *functionals* $\mathcal{C}(x, \bar{q}(z,t)$ such that $\delta C(x, \bar{q}(z,t)$ belongs to the space

$$\begin{aligned}
P_{Cas} = \quad & \{e_1, e_{pq} \mid \exists \mathcal{DR}_C \text{ s.t } \exists e_2 : \\
& (0, e_1, -f_{R_1}, e_{R_1}, 0, e_{pq}, -f_{Rpq}, e_{Rpq}, 0, e_2, -f_{R_2}, e_{R_2}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C\},
\end{aligned} \tag{4.54}$$

with (f_{R1}, e_{R1}) denoting the flows and efforts variables of the dissipation term in the finite-dimensional part of the plant Dirac structure, and (f_{R2}, e_{R2}) the flow and effort variables associated with the dissipation term in the controller Dirac structure (finite-dimensional).

Similar to the finite-dimensional case, the following theorem addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state by characterization of the space P_{Cas}.

Proposition 4.22. *The space P_{Cas} defined above is equal to the space*

$$\tilde{P} = \{e_1, e_{pq} \mid \exists (f_b, e_b) \text{ s.t } (0, e_1, f_{R1}, e_{R1}, 0, e_{pq}, -f_{Rpq}, e_{Rpq}, f_b, e_b) \in \mathcal{DR}_P\}.$$

where (f_b, e_b) are the boundary variables

Proof. The inclusion $\tilde{P} \subset P_{Cas}$ is again obtained by taking $\mathcal{DR}_2 = \mathcal{DR}_1^*$ □

Example 4.23. A simple example in this case would be to consider a plant system where we interconnect the transmission line at one end to a finite-dimensional port-Hamiltonian system, the Dirac structure of which would be given as

$$\begin{bmatrix} -\dot{x}_1 \\ -\partial_t q \\ -\partial_t \phi \end{bmatrix} = \left(\begin{bmatrix} -J(x) & 0 & 0 \\ 0 & 0 & \mathrm{d} \\ 0 & \mathrm{d} & 0 \end{bmatrix} + \begin{bmatrix} R(x) & 0 & 0 \\ 0 & G* & 0 \\ 0 & 0 & R* \end{bmatrix} \right) \begin{bmatrix} \partial_x H \\ \delta_q H \\ \delta_\phi H \end{bmatrix} - \begin{bmatrix} g(x) \\ 0 \\ 0 \end{bmatrix} \delta_q H \mid_0$$
$$\begin{bmatrix} f_l \\ e_l \end{bmatrix} = \begin{bmatrix} -\delta_\phi H \mid_l \\ \delta_q H \mid_l \end{bmatrix}$$
$$\delta_\phi H \mid_0 = g^T(x) e_1.$$

The achievable Casimirs in this case are all functionals $\mathcal{C}$ such that

$$\begin{aligned} &J(x)\partial_x \mathcal{C} + g(x)\delta_q \mathcal{C} \mid_0 = 0 \\ &g_R^T(x)\partial_x \mathcal{C} = 0 \\ &\mathrm{d}\delta_\phi \mathcal{C} - G * \delta_q \mathcal{C} = 0 \\ &\mathrm{d}\delta_q \mathcal{C} - R * \delta_\phi \mathcal{C} = 0. \end{aligned}$$

We see that the first two conditions are the same as that obtained for the finite-dimensional case (4.35) and the last two conditions are those corresponding to the transmission line (4.50). It is easily seen that these conditions are a combination of those obtained for the finite-dimensional and the infinite-dimensional systems respectively.

Remark 4.24. In a similar way one can also consider other mixed cases, where the plant system is an infinite-dimensional system and the controllers being finite-dimensional systems interconnected through the boundary of the infinite-dimensional plant system. This is the case then we wish to control an infinite-dimensional system, with finite-dimensional systems through the boundary. We consider such a case while dealing with control of fluid dynamical systems in Chapter 5.

5

Control of port-Hamiltonian systems

"Things should be made as simple as possible, but not any simpler." - Albert Einstien.

5.1 Control of finite-dimensional systems

In the previous chapters we have seen that a key feature of port-Hamiltonian systems is that the power-conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system, with total state space the product of the state spaces of the components, total Hamiltonian being the sum of the Hamiltonian functions and the Dirac structure defined by the composition of the Dirac structures of the subsystems. We have derived formulas for the composition of Dirac structures and the set of achievable Dirac structures (by composition of a given plant Dirac structure with a to-be-designed controller Dirac structure). We have also seen how this leads to a characterization of the set of achievable Casimir functions.

In this chapter we discuss how to exploit these properties of port-Hamiltonian systems for control purposes. We are basically interested in the control problem of set point regulation, using energy shaping techniques. We focus on how to use the results obtained on achievable Casimirs for the closed-loop systems in Chapter 4 for analyzing the stability of the closed-loop system. We use the Casimirs in the extended state space to generate Lyapunov functions of the closed-loop system as the sum of the plant and the controller Hamiltonians and the resulting Casimir function. We also see how by generating Casimirs in the extended state space, we overcome the problem of initializing the controller arising in the general control by interconnection method. Towards the end we study the limitations of this method and see how by the use of new passive outputs we generate Lyapunov functions for a class of forced Hamiltonian systems with dissipation. We also briefly highlight the IDA-PBC method, which allows energy shaping by modification of the interconnection and damping matrices and how it overcomes certain drawbacks of the control by interconnection methodology.

5.1.1 Energy-balancing control

In this section we are interested in finite-dimensional port-Hamiltonian systems in the input-output form as

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ are the energy variables, the smooth function $H(x) : \mathbb{R}^n \to \mathbb{R}$ represents the total energy and $u, y \in \mathbb{R}^m$ are the port power variables. The port variables u and y are conjugated variables, in the sense that their duality product defines the power flows exchanged with the environment of the system. The interconnection structure is captured in the $n \times n$ skew-symmetric matrix $J(x) = -J^T(x)$ and the $n \times m$ matrix $g(x)$, while $R(x) = R^T(x) \geq 0$ represents the dissipation, all these matrices depend smoothly on the state x. The port-Hamiltonian system in the input-output form then satisfies the energy balance equation

$$\underbrace{H[x(t)] - H[x(0)]}_{stored\ \ energy} = \underbrace{\int_0^t u^\top(\tau)y(\tau)d\tau}_{supplied} - \underbrace{\int_0^t \frac{\partial^T H}{\partial x}(x(\tau))R(x(\tau))\frac{\partial H}{\partial x}(x(\tau))d\tau}_{dissipated}\,. \tag{5.2}$$

This implies that port-Hamiltonian systems are passive with respect to the supply rate $u^T y$ if the Hamiltonian is non-negative (or, bounded from below).

The control objective is to regulate the static behavior, that is, the equilibria, which is determined by the shape of the energy function. It is therefore natural to recast our control problem in terms of finding a dynamical system and an interconnection pattern such that the overall energy function takes the desired form. There are at least two important advantages of adopting such an "energy shaping"perspective of control:

1. The energy function determines not just the static behavior, but also, via the energy transfer between subsystems (through the ports), its transient behavior. Focusing our attention on the systems energy, we can then aim, not just at stabilization, but also at *performance* objectives that can, in principle, be expressed in terms of "optimal"energy transfer. Performance and not stability is, of course, the main concern in applications.

2. Practitioners are familiar with energy concepts, which can serve as a *lingua franca* to facilitate communication with control theorists, incorporating prior knowledge and providing physical interpretations of the control action.

Passivity–based control techniques (PBC) [38, 58] achieve stabilization of nonlinear *feedback passive* systems assigning a storage function with a minimum at the desired equilibrium. For physical systems a natural candidate storage function is the difference between the stored and the supplied energies, leading to the so-called *energy–balancing control*, whose underlying stabilization mechanism is particularly appealing. Two important corollaries follow from (5.2)

- The energy of the uncontrolled system (i.e., with $u \equiv 0$) is nonincreasing (that is, $H[x(t)] \leq H[x(0)]$), and it will actually decrease in the presence of dissipation. If the energy function is bounded from below, the system will eventually stop at a point of minimum energy. Also, as expected, the rate of convergence of the energy function is increased if we extract energy from the system, for instance, setting $u = -K_{di}y$, with $K_{di} = K_{di}^\top > 0$ a so–called damping injection gain.
- Given that
$$-\int_0^t u^\top(s)y(s)ds \leq H[x(0)] < \infty,$$
the total amount of energy that can be extracted from a passive system is bounded. [This property, which (somehow misleadingly) is often stated with the inequality inversed, will be instrumental in identifying the class of systems that are stabilizable with energy balancing PBC.]

Often, the point where the open–loop energy is minimal (which typically coincides with the zero state) is not the one of interest, and control is introduced to operate the system around some nonzero equilibrium point, say $x_\star$. Hence, the control problem consists in finding a control input $u = \beta(x) + v$ such that the energy supplied by the controller can be expressed as a function of the state. Indeed, from (5.2) we see if we can find a function $\beta(x)$ satisfying

$$-\int_0^t \beta^\top[x(s)]y(s)ds = H_a[x(t)] + \kappa,$$

for some function $H_a(x)$, then the control $u = \beta(x) + v$ will ensure that the map $v \to y$ is passive with new energy function

$$H_d(x) = H(x) + H_a(x). \tag{5.3}$$

For port–Hamiltonian systems, the following proposition characterizes the class of functions $\beta(x)$ and $H_a(x)$ such that the closed–loop system satisfies the new energy–balancing equation

$$H_d[x(t)] - H_d[x(0)] = \int_0^t v^\top(\tau) y(\tau) d\tau - \int_0^t \frac{\partial^T H_d}{\partial x}(x(\tau)) R(x(\tau)) \frac{\partial H_d}{\partial x}(x(\tau)) d\tau,$$

with the dissipation term $d_d(t) \geq 0$ to increase the convergence rate.

Proposition 5.1. *[58] Consider the port–Hamiltonian system (5.1), if we can find a function $\beta(x)$ and a vector function $K(x)$ satisfying*

$$[J(x) - R(x)]K(x) = g(x)\beta(x),$$

such that

i) $\frac{\partial K}{\partial x}(x) = \left(\frac{\partial K}{\partial x}(x)\right)^T$,

ii) $K(x_\star) = -\frac{\partial H}{\partial x}(x_\star)$,

iii) $\frac{\partial K}{\partial x}(x_\star) > -\frac{\partial^2 H}{\partial x^2}(x_\star)$.

Then the closed–loop system is a port-Hamiltonian system of the form

$$\dot{x} = [J(x) - R(x)]\frac{\partial H_d}{\partial x}(x), \tag{5.4}$$

with H_d given by (5.3) and H_a satisfying $K = \frac{\partial H_a}{\partial x}(x)$. Furthermore, $x_\star$ is an stable equilibrium point of (5.4).

The control law $u = \beta(x)$ is customarily called a passivity-based control law, since it is based on the passivity properties of the original system (5.1) and transforms it into another passive system with shaped storage function H_d.

The dissipation obstacle

Unfortunately, energy–balancing stabilization is stymied by the existence of pervasive dissipation, that appears in many engineering applications. Indeed, it has been shown in [40] that a necessary condition to satisfy Proposition 5.1 is that the natural damping of the port–Hamiltonian system satisfies

$$R(x_*)K(x_*) = 0.$$

If $R(x)$ is diagonal, this condition requires that no damping is present in the coordinates that need to be shaped, that is, the coordinate where the function $H(x)$ has to be modified. To characterize the dissipation obstacle it is convenient to adopt a *control–by–interconnection* viewpoint, which clearly reveals the limitations of energy–balancing control, as we will see in the following sections. Some control methodologies as the interconnection and damping assignment passivity–based control and power shaping, has been proposed in [40] and [37] respectively to overcome the dissipation obstacle. See also Section 5.1.4 for further details.

5.1.2 Control by interconnection

In this section, we use the results on achievable Casimirs obtained in Chapter 4 to actually solve control problems involving stabilization of a given system around a desired equilibrium. In Section 4.2 we argued how by generating Casimir functions we can stabilize the closed-loop system in the case H_P does not have a minimum at the desired equilibrium. The resulting Lyapunov function is given by the sum of Hamiltonians of the plant and the controller systems and the corresponding Casimir function. We first recall the general theory of control by interconnection in which we restrict the motion of the system to a subspace of the extended state space. Next, we extend the method of control by interconnection by making use of achievable Casimirs in the extended state space.

General theory

The general theory on control by interconnection of port-Hamiltonian systems relies on the generation of Casimirs for the closed-loop system by looking at the level sets of the Casimirs as *invariant submanifolds* of the combined plant and controller state space $\mathcal{X}_p \times \mathcal{X}_c$. Restricted to every such invariant submanifold (part of) the controller state can be expressed as a function of the plant state, whence the closed-loop Hamiltonian restricted to such an invariant manifold can be seen as a *shaped* version of the plant Hamiltonian. To be explicit suppose that we have found Casimirs of the form

$$\xi_i - F_i(x),\ i = 1, \cdots, n_p,$$

where n_p is the dimension of the controller state space, then on every invariant manifold $\xi_i - F_i(x) = \alpha_i, i = 1, \cdots, n_p$, where $\alpha = (\alpha_1, \cdots, \alpha_{n_p})$ is a vector of constants depending on the initial plant and controller state, the closed-loop Hamiltonian can be written as

$$H_s(x) := H_p(x) + H_c(F(x) + \alpha),$$

where, as before, the controller Hamiltonian H_c can still be assigned. This can be regarded as *shaping* the original plant Hamiltonian H_p to a new Hamiltonian H_s.

Example 5.2. Consider the equations of a normalized pendulum

$$\ddot{q} + \sin q + d\dot{q} = u,$$

with d a positive damping constant. The total energy is given by $H(q, p) =$

$\frac{1}{2}p^2 + (1 - \cos q)$. The corresponding Dirac structure is given as

$$\begin{bmatrix} -\dot{q} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sin q \\ p \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ dp \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$e_R = \begin{bmatrix} 0 & p \end{bmatrix}$$
$$e_p = p.$$

The elements of the Dirac structure are given by

$$(f_p, e_p, f_R, e_R, f, e) = \left(-[\dot{q}, \dot{p}]^T, [\sin q, p]^T, [0\ dp]^T, [0\ p]^T, u, e\right)$$
$$g_R(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\ g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case the achievable Casimirs (in terms of the plant state (q, p)) are such that

$$\frac{\partial C}{\partial p} = 0.$$

The above expression implies that any Casimir function for this system does not depend on the p term, which is precisely where dissipation enters into the system. However we can find Casimirs depending on q and can use it for stability analysis as shown below.

Stability analysis: Let $q_\star$ be a desired equilibrium position of the pendulum. The objective is to shape the potential energy $P(q) = 1 - \cos q$ in such a way that it has a minimum at $q = q_\star$. Consider a first order controller written in the input output form as

$$\dot{\xi} = u_c$$
$$y_c = \frac{\partial H_c}{\partial \xi},$$

the corresponding elements of the Dirac structure being

$$(f_c, e_c, f^{'}, e^{'}) = (\dot{\xi}, \frac{\partial H_c}{\partial x_c}, u_c, y_c).$$

The interconnection constraints between the plant and the controller are given as

$$u = y_c, \quad u_c = -y_p. \tag{5.5}$$

We know from Chapter 4 that the Casimirs of the closed-loop system are functions $C(x, \xi)$ such that

$$(0, \frac{\partial C}{\partial x}, 0, 0, 0, \frac{\partial C}{\partial \xi}) \in \mathcal{D}, \tag{5.6}$$

also see Equation (4.34). Since we are looking for Casimirs of the form $C = \xi - F(q)$, the solution to (5.6) are functions of the form $F(q) = q$. Choosing

$$P_c(\xi) = \cos\xi + \frac{1}{2}(\xi - q_\star)^2,$$

and substituting $\xi = F(q) + c = q + c$ we get the shaped potential energy as

$$\begin{aligned} P_d(q) &= P(q) + P_c(F(q) + c) \\ &= (1 - \cos q) + \cos(q + c) + \frac{1}{2}(q + c - q_\star)^2. \end{aligned}$$

However, in order to obtain a minimum at $q = q_*$ the controller needs to be initialized in such a way that $c = 0$. The passivity based control $u = \alpha(x)$ is simply

$$u = -\frac{\partial H_c}{\partial \xi}(q) = \sin q - (q - q_*).$$

Casimirs in the extended state-space

In this section a modification of the control by interconnection method to overcome the problem of controller initialization mentioned above is proposed. The key idea is to analyze the closed-loop system in the extended state space $\mathcal{X} \times \mathcal{X}_c$, with the control objective of stabilization of a desired equilibrium (x_*, ξ_*), for some ξ_* satisfying equilibrium conditions of the closed-loop system. To this end, we consider general Casimir functions $C : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$. These can be characterized by the space as in (5.6). In this case we see that we can not only use $C(x, \xi)$ as a Casimir function but all functions of the form $\Psi(C(x, \xi))$ can be used for the purpose of the stability analysis of the closed-loop system. Thus we have a whole family of Casimirs to choose from, instead of specific Casimirs. On the basis of the Hamiltonian of the plant, the Hamiltonian of the controller and the corresponding Casimir function a Lyapunov function candidate is built as the sum of the plant and controller Hamiltonians and the Casimir function as

$$V(x,\xi) = H(x) + H_c(\xi) + \Psi(C(x, \xi)). \tag{5.7}$$

The time derivative of the Lyapunov function then satisfies

$$\frac{d}{dt}V(x, \xi) = -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial^T H}{\partial x}(x) - \frac{\partial^T H_c}{\partial \xi}(\xi)R_c(\xi)\frac{\partial^T H_c}{\partial \xi}(\xi) \leq 0,$$

and hence $V(x, \xi)$ qualifies as a Lyapunov function for the closed-loop dynamics.

The next step would be to shape the closed-loop energy in the extended state space (x, ξ) in such a way that it has a minimum at $(x_\star, \xi_*)$. (Note that x_* is the desired equilibrium point of the plant system, while the controller equilibrium point ξ_* can be chosen arbitrarily). Therefore we require that the gradient of (5.7) has an extremum at $(x_\star, \xi_*)$ and that the Hessian at $(x_\star, \xi_*)$ is positive definite, that is

$$\begin{bmatrix} \frac{\partial}{\partial x}[H(x) + \Psi(C(x,\xi))] \mid_{(x_\star,\xi_*)} \\ \frac{\partial}{\partial \xi_c}[H_c(\xi) + \Psi(C(x,\xi_c))] \mid_{(x_\star,\xi_*)} \end{bmatrix} = 0, \tag{5.8}$$

and

$$\left.\begin{bmatrix} \frac{\partial^2}{\partial x^2}[H(x) + \Psi(C(x,\xi))] & \frac{\partial^2}{\partial \xi \partial x}\Psi(C(x,\xi))] \\ \frac{\partial^2}{\partial x \partial \xi}\Psi(C(x,\xi))] & \frac{\partial^2}{\partial x_c^2}[H_c(x_c) + \Psi(C(x,\xi))] \end{bmatrix}\right|_{(x_\star,\xi_\star)} \geq 0. \tag{5.9}$$

Suppose that $V(x,\xi)$ has a strict local minimum at $(x_\star, \xi_*)$, that is, there exists an open neighborhood $\mathcal{B}$ of $(x_\star, \xi_*)$ $V(x,\xi) > V(x_*, \xi_*)$ for all $x \in \mathcal{B}$. Furthermore assume that the largest invariant set under the closed–loop dynamics contained in

$$\begin{aligned} \{(x,\xi) \in \mathcal{X} \times \mathcal{X}_c \cap \mathcal{B} | \frac{\partial^\top H}{\partial x}(x) R(x) \frac{\partial^\top H}{\partial x}(x) = 0, \\ \frac{\partial^\top H_c}{\partial \xi}(\xi) R_c(\xi) \frac{\partial^\top H_c}{\partial \xi}(\xi) = 0\}, \end{aligned}$$

equals $(x_\star, \xi_*)$, with ξ_* being arbitrary. Then $(x_\star, \xi_*)$ is a locally asymptotically stable equilibrium of the closed–loop system.

Example 5.3. Consider a mechanical system with damping and actuated by external forces u described as port-Hamiltonian system with dissipation

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= (\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D(q) \end{bmatrix}) \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \\ y &= B^T(q) \frac{\partial H}{\partial p}, \end{aligned} \tag{5.10}$$

with $x = \begin{bmatrix} q \\ p \end{bmatrix}$, where $q \in \mathbb{R}^n$ are the generalized configuration coordinates, $p \in \mathbb{R}^n$ the generalized momenta, and $D(q) = D^T(q) \geq 0$ is the damping matrix. If $D(q) > 0$, then it is said that the system is fully damped. The outputs $y \in \mathbb{R}^m$ are the generalized velocities corresponding to the generalized external forces $u \in \mathbb{R}^m$. In most cases the Hamiltonian $H(q,p)$ takes the form

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + P(q), \tag{5.11}$$

where $M(q) = M^T(q) > 0$ is the generalized inertia matrix, $\frac{1}{2} p^T M^{-1}(q) p = \frac{1}{2} \dot{q}^T M(q) \dot{q}$ is the kinetic energy, and $P(q)$ is the potential energy of the system.

Consider now a controller port-Hamiltonian system

$$
\begin{aligned}
\dot{\xi} &= [J_c(\xi) - R_c(\xi)]\frac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c \\
y_c &= g_c^T(\xi)\frac{\partial H_c}{\partial \xi}(\xi).
\end{aligned}
$$

Then the equations (5.6) for $C = (C_1(x,\xi), ..., C_m(x,\xi))^T$ take the form

$$
\frac{\partial^T C}{\partial p}\frac{\partial C}{\partial q} - \frac{\partial^T C}{\partial q}\frac{\partial C}{\partial p} = \frac{\partial^T C}{\partial \xi}J_c(\xi)\frac{\partial C}{\partial \xi}
$$

$$
D(q)\frac{\partial C}{\partial p} = 0 = R_c(\xi)\frac{\partial C}{\partial \xi}
$$

$$
\frac{\partial^T C}{\partial p} = 0, \text{ and } \frac{\partial^T C}{\partial q} = -\frac{\partial^T C}{\partial \xi}g_c(\xi)B^T(q),
$$

or equivalently

$$
J_c = 0, \quad \frac{\partial C}{\partial p} = 0, \quad \frac{\partial^T C}{\partial q} + \frac{\partial^T C}{\partial \xi}g_c(\xi)B(q) = 0. \tag{5.12}
$$

Hence if we can solve the PDE in the above equation, then the closed-loop port-Hamiltonian system with $J_c = 0$ admits Casimirs $C_i(x,\xi)$, $i = 1, ..., m$. leading to a closed loop system

$$
\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\xi} \end{bmatrix} &= \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & -B(q)g_c^T(\xi) \\ 0 & g_c(\xi)B(q) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial q} \\ \frac{\partial H_c}{\partial p} \\ \frac{\partial H_c}{\partial \xi} \end{bmatrix} \\
y &= B^T(q)\frac{\partial H_c}{\partial x} \\
y_c &= g_c(\xi)\frac{\partial H_c}{\partial \xi},
\end{aligned}
$$

for the shaped system

$$
H_s(q,p,\xi) = H(q,p) + H_c(\xi).
$$

If $H(q,p)$ is given as in (5.11), then

$$
H_s(q,p,\xi) = \frac{1}{2}p^T M(q)p + P(q) + H_c(\xi),
$$

and the control amounts to shaping the potential energy of the system.

Example 5.4. We again consider the case of the normalized pendulum as in the above example. With first order controllers, we need to solve (5.12) for Casimirs of the closed loop system. In other words we are looking for the solution of the pde $\frac{\partial C}{\partial x}(x,\xi) = \frac{\partial C}{\partial \xi}(x,\xi)$. The solution of this pde should be of the form $C(x,\xi) = q - \xi$, hence any function of the form $\Psi(q-\xi)$ is a Casimir for the closed-loop system.
Stability Analysis: The objective is to stabilize the system at a desired equilibrium in the extended state space $(q_\star, \xi_*)$. We shape the potential energy in such a way that it has a minimum at $q = q_\star, \xi = \xi_*$. This can be achieved by choosing a controller Hamiltonian of the form

$$H_c(\xi) = \frac{1}{2}\beta(\xi - \xi_* - \frac{1}{\beta}\sin q_\star)^2,$$

and the function $\Psi(C(q,\xi)) = \Psi(q-\xi)$ as

$$\Psi(q-\xi) = \frac{1}{2}k(q - q_\star - (\xi - \xi_*) - \frac{1}{k}\sin q_\star)^2,$$

where β and k are chosen to satisfy (5.8) and (5.9). Simple computations show that β and k should be chosen such that

$$\cos q_\star + k > 0,\ \beta\cos q_\star + k\cos q_\star + k\beta > 0.$$

The resulting passivity based input u, which is a dynamic output feedback, is then given by

$$u = -\frac{\partial H_c}{\partial \xi}(\xi) = -\beta(\xi - \xi_* - \frac{1}{\beta}\sin q_\star).$$

Remark 5.5. In the same way we can also stabilize a system of n "fully actuated"pendulums connected to each other, in which case we have to solve n different pde's for each of the subsystem, in order to find the corresponding Casimir functions.

5.1.3 Passivity with respect to a new output

As discussed in section 4.2 of the previous chapter, there are cases of systems (for example the case of a parallel RLC circuit in Example 4.11 or the special case of the capacitor microphone as in Example 2.9) where we cannot find Casimirs for the closed-loop system and hence cannot apply the control by interconnection method for stabilizing a given plant system. So far in our analysis and in the interconnection of the plant system with the controller system,we have taken the "natural "port variables to define the passive map. Some recent work [21], has established the existence of alternative passive maps for port-Hamiltonian systems. We now see how with these new port-variables the interconnected system admits new dynamical invariants and how can we can use them for energy shaping. We state here, without proof, the following proposition:

Proposition 5.6. *[21] Consider the input state output port-Hamiltonian system (5.1). Assume that $[J(x) - R(x)]$ is full rank. The the system satisfies the new energy balance equality*

$$\frac{dH}{dt} \leq \int u^T \tilde{y},$$

where $\tilde{y} = \tilde{h}(x, u)$, with

$$\begin{aligned}\tilde{h}(x,u) &= -g^T(x)(J(x) - R(x))^{-T}\left\{(J(x) - R(x)\tfrac{\partial H}{\partial x} + g(x)u\right\} \\ &= -g^T(x)(J(x) - R(x))^{-T}\dot{x}.\end{aligned}$$

Hence, if $H(x)$ is bounded from below, the system is passive with respect to the supply-rate $u^T\tilde{y}$ and storage function $H(x)$.

Remark 5.7. From the above proposition we have

$$\dot{H}(x) = -\dot{x}^T R(x)\dot{x} + u^T\tilde{y},$$

where $\dot{x} \triangleq (J(x) - R(x))^{-T}\dot{x}$. Comparing with the classical power balance equation,

$$\dot{H}(x) = -\frac{\partial^T H}{\partial x} R(x) \frac{\partial H}{\partial x} + u^T y.$$

This implies that the new passivity property is established by "swapping the damping".

We make use of this new passivity property to study stability of forced Hamiltonian system with dissipation [29], i.e. to analyze stability of the system for a constant but nonzero input leading a forced equilibrium $\bar{x} \in \mathcal{X}$. Corresponding to $u = \bar{u}$, the forced equilibria are solutions of

$$[J(\bar{x}) - R(\bar{x})]\frac{\partial H}{\partial x}(\bar{x}) + g(\bar{x})\bar{u} = 0.$$

In general, a forced equilibrium will not be a minimum (or an extremum) of H. Furthermore inserting $u = \bar{u}$ into the *new* energy balance equation yields

$$\frac{dH}{dt} = -\dot{x}^T R(x)\dot{x} + \bar{u}^T\tilde{y}(x), \tag{5.13}$$

having a right hand side, which in general will be non-positive. Thus in most cases the Hamiltonian cannot be directly used as a Lyapunov function for stability of the forced equilibrium $\bar{x}$. One way of approaching the problem is to start from the power balance of the forced system (5.13) and to bring the second term on the right-hand side to the left hand side, suggesting to look for candidate Lyapunov functions

$$H(x(t)) - \bar{u}^T \int_0^t \tilde{y}(\tau)d\tau. \tag{5.14}$$

To check whether (5.13) can be used as a Lyapunov function, the first basic question is if we can write $\bar{u}^T \int_0^t \tilde{y}(\tau)d\tau$ as a function of the state x. From a control theoretic point of view, this suggests to consider a cascade of Σ with input $\bar{u}$, followed by integration of y and to look for Lyapunov functions of the composed system

$$\dot{x} = [J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + g(x)\bar{u}$$

$$\dot{\xi} = \tilde{y}(x,u), \quad \xi \in \mathbb{R}^m.$$

This can be written as an unforced Hamiltonian system with dissipation

$$\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix} = \begin{bmatrix}[J(x)-R(x)] & -g(x)\\ -g^T(x)[J(x)-R(x)]^{-T}[J(x)-R(x)] & g^T(x)[J(x)-R(x)]^{-T}g(x)\end{bmatrix}\begin{bmatrix}\frac{\partial H_a}{\partial x}\\ \frac{\partial H_a}{\partial \xi}\end{bmatrix}, \tag{5.15}$$

with $H_a(x,\xi)$ the augmented energy function

$$H_a(x,\xi) \triangleq H(x) + H_s(\xi), \quad H_s(\xi) \triangleq -\bar{u}^T\xi \ .$$

Writing $\bar{u}^T \int_0^t \tilde{y}(\tau)d\tau$ as a function of $x(t)$ then corresponds to expressing $\xi(t)$ as a function of $x(t)$ along the dynamics (5.13). This in turn is true, if there exist Casimirs of the form

$$C_i(x,\xi) = \xi_j - F_j(x), \ i \in m.$$

We then look for Lyapunov function candidates of the form (see [29] for details)

$$\begin{aligned}\mathcal{V}(\chi) &= H(x) + H_c(\xi), \ \ H_c(\xi) = -\bar{u}^T\xi\\ &= H(x) - \sum_{j=1}^{m} \bar{u}_j(C_j(x) + c).\end{aligned}$$

Example 5.8 (Example 4.11 continued). We had seen before in Chapter 4 that there do not exist Casimirs for the parallel RLC circuit with the standard passive output. We now discuss how, with the help of new passive outputs we can achieve desired stability. The new passive output in this case would be $\frac{\dot{\phi}}{R} + \dot{q}$, where $\dot{\phi}$ is the voltage in the inductor and $\dot{q}$ is the current through the capacitor. Compare with the original output which is just $\frac{\phi}{L}$, the current through the inductor. The closed-loop dynamics (5.15) take the form

$$\begin{bmatrix}\dot{q}\\ \dot{\phi}\\ \dot{\xi}\end{bmatrix} = \begin{bmatrix}-1/R & 1 & 0\\ -1 & 0 & -1\\ -2/R & 1 & 1/R\end{bmatrix}\begin{bmatrix}\frac{\partial H_a}{\partial q}\\ \frac{\partial H_a}{\partial \phi}\\ \frac{\partial H_c}{\partial \xi}\end{bmatrix},$$

with $H_a(x,\xi) = H(x) - \bar{u}\xi$. The closed-loop system is the same as that obtained in [29], where the plant system was "embedded"into a larger system. The system then admits Casimirs of the form

$$C(x,\xi) = q + \frac{1}{R}\phi - \xi.$$

The corresponding Lyapunov function is then given by

$$\mathcal{V}(\chi) = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}\frac{\phi^2}{L} - \bar{u}\left(q + \frac{1}{R}x_2\right) + \frac{\bar{u}^2}{2}\left(C_1 + \frac{L}{R^2}\right).$$

Example 5.9 (Example 2.9 continued). Consider the capacitor microphone with the new outputs which are $(\dot{q}, \dot{Q})$. Comparing this with the standard outputs, we have

$$\begin{aligned}\tilde{y}_1 &= y_1\\ \tilde{y}_2 &= y_2 + \frac{1}{R}E.\end{aligned}$$

The control objective is to stabilize the system at the equilibrium point given by $(\bar{q}, 0, \bar{Q})$, with

$$\frac{\bar{q}\bar{Q}}{A\epsilon} = \bar{E}, \bar{Q}^2 = 2A\epsilon\bar{F},$$

where $\bar{E}$ and $\bar{F}$ are the constant inputs. Interconnecting it to a controller of the form

$$\begin{aligned}\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} &= \begin{bmatrix}u_1\\ u_2\end{bmatrix}\\ \begin{bmatrix}y_{c_1}\\ y_{c_2}\end{bmatrix} &= \begin{bmatrix}\frac{\partial H_c}{\partial \xi_1}\\ \frac{\partial H_c}{\partial \xi_2}\end{bmatrix}.\end{aligned}$$

This yields the following closed-loop system dynamics

$$\begin{bmatrix}\dot{q}\\ \dot{p}\\ \dot{Q}\\ \dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 & 0\\ -1 & -c & 0 & -1 & 0\\ 0 & 0 & -\frac{1}{R} & 0 & -\frac{1}{R}\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{R} & 0 & -\frac{1}{R}\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\\ \frac{\partial H}{\partial Q}\\ \frac{\partial H_c}{\partial \xi_1}\\ \frac{\partial H_c}{\partial \xi_2}\end{bmatrix}.$$

It can easily be seen that the closed-loop systems admits Casimirs of the form

$$-\xi_1 - \xi_2 + x_1 + x_3 + c.$$

We then have the following Lyapunov function

$$\mathcal{V}(\mathcal{X}) = \frac{1}{2m}p^2 + \frac{1}{2}k(q-\bar{q})^2 + \frac{q}{2A\epsilon}Q^2 - \bar{F}q - \bar{E}Q + \bar{E}\sqrt{2A\epsilon\bar{F}}.$$

Of course, the mere existence of Casimirs is not a sufficient condition for achieving the desired stability properties. This is due to the fact that the Casimirs, even though they exist, need not necessarily be a function of the coordinate which we wish to *shape*. Below we present an example of a system where we can find Casimirs for the closed-loop system but still cannot achieve the desired stability properties.

Example 5.10. The model of a permanent magnet synchronous machine [48], in the case of an isotropic rotor, in the dq frame can be written as a port-Hamiltonian system in the input state output form, with the state vector $x = [x_1, x_2, x_3]^\top$ and

$$J(x) = \begin{bmatrix} 0 & \frac{LP}{J}x_3 & 0 \\ -\frac{LP}{J}x_3 & 0 & -\phi \\ 0 & \phi & 0 \end{bmatrix},$$

$$R(x) = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where x_1, x_2 are the stator currents, x_3 is the angular velocity, P is the number of pole pairs, L is the stator inductance, R_s is the stator winding resistance, and Φ and J are the dq back emf constant and the moment of inertia both normalized with P. The inputs are the stator voltages $[v_d,\ v_q]^\top$. The energy function of the system is given by

$$H(x) = \frac{1}{2}\left(Lx_1^2 + Lx_2^2 + \frac{J}{P}x_3^2\right).$$

The desired equilibrium to be stabilized is usually selected based on the so–called “maximum torque per ampere” principle as $x_\star = [0,\ \frac{L\tau_l}{P\Phi},\ \frac{J}{P}x_{3\star}]^\top$ where τ_l is the constant load torque.[1]

Interconnecting the plant system with a port-Hamiltonian control

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix}$$

$$\begin{bmatrix} y_{c1} \\ y_{c2} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_c}{\partial \xi_1}(\xi_1, \xi_2) \\ \frac{\partial H_c}{\partial \xi_2}(\xi_1, \xi_2) \end{bmatrix},$$

[1] In the port-Hamiltonian modeling of the permanent magnet synchronous machine, τ_l acts as a perturbation to the system.

via the power preserving interconnection

$$\begin{array}{ll} v_d = -y_{c1}, & v_q = -y_{c2}, \\ u_{c1} = \frac{\partial H}{\partial x_1}(x), & u_{c2} = \frac{\partial H}{\partial x_2}(x) \end{array},$$

yields the closed–loop system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -R_s & \frac{LP}{J}x_3 & 0 & -1 & 0 \\ -\frac{LP}{J}x_3 & -R_s & -\phi & 0 & -1 \\ 0 & \phi & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \\ \frac{\partial H_c}{\partial \xi_1} \\ \frac{\partial H_c}{\partial \xi_2} \end{bmatrix}. \tag{5.16}$$

Solving for Equation (5.6) we get that the Casimir function is given by $C = \frac{1}{\Phi}x_3 - \xi_2$. Thus, the resulting Lyapunov function would be of the form (5.7)

$$V(x,\xi) = \frac{1}{2}\left(Lx_1^2 + Lx_2^2 + \frac{J}{P}x_3^2\right) + H_c(\xi) + \Psi(\frac{1}{\Phi}x_3 - \xi_2).$$

However, we can see that the equilibrium assignment condition (5.8) cannot be satisfied, because we need to shape both x_2 and x_3 to assign $x_\star$, and the Casimir depends only on x_3.

In general, it is not possible to apply the Control by Interconnection methodology to the family of electromechanical systems described in [50]. Firstly, in most cases, the closed–loop matrix $J_{cl}(x,\xi) - R_{cl}(x,\xi)$ is full–rank, leading to Casimir functions of the form $C(x,\xi) = c$, with c a constant vector, which obviously cannot be used to shaped the energy of the system. Secondly, even if we can determine the Casimirs—as in the case of the permanent magnet synchronous machine—, these functions do not depend on the coordinates we need to shape. The source of the problem is the lack of interconnection between the electrical and mechanical subsystems.

Also in the case of mechanical systems the control law generated by the control by interconnection method amounts to shaping the potential energy of the system. However, in many cases, for example in the case of underactuated mechanical systems, we need to shape the kinetic energy of the system or in other words modify the interconnection structure of the system. This is clearly not possible by this control strategy. To overcome this limitation a control strategy called Interconnection and damping assignment passivity based control has been proposed in [40], which enables us to modify the interconnection structure of the system and also overcome the dissipation obstacle. We briefly highlight this method here.

5.1.4 Interconnection and damping assignment passivity based control (IDA-PBC)

IDA–PBC was introduced in [40] as a procedure to control physical systems described by port-Hamiltonian models in input–output form as in (5.1). The idea is to generate a state-feed back control law which enables us to regulate the behavior of the nonlinear system by assigning a desired interconnection and damping structures to the closed-loop. If we talk in terms of a mechanical system, IDA-PBC enables us not only to shape the potential energy, but also the kinetic energy. This is in contrast to the control by interconnection method, where the control law for a mechanical system amounts to shaping the potential energy of the system, see Example 5.3. The IDA-PBC method also enables us to shape the energy without the generation of Casimirs.

In the IDA–PBC procedure we select the structure of the closed–loop system as another port-Hamiltonian system and then we characterize all assignable energy functions compatible with this structure. This characterization is given in terms of the solution of a partial differential equation (PDE) which is parameterized by three (designer chosen) matrices that are related with the interconnection between the subsystems, the damping and the kernel of the systems input matrix, respectively. Several interpretations can be given to the role played by these matrices. At the most basic—computational—level they can be simply viewed as degrees–of–freedom to simplify the solution of the PDE. In the case of physical systems the interconnection and the damping matrices determine the energy exchange and the dissipation of the system, respectively, consequently they can often be judiciously chosen invoking this kind of physical considerations. See [36] for an extensive list of references and applications of this methodology.

The main proposition of IDA–PBC for port–Hamiltonian systems is stated as follows

Proposition 5.11. *[40] Consider the system (5.1), assume there are matrices $g^{\perp}(x)$, $J_d(x) = -J_d^T(x)$, $R_d(x) = R_d^T(x) \geq 0$ and a function $H_d : \mathbb{R}^n \to \mathbb{R}$ that verify the PDE*

$$g^{\perp}(x)[J(x) - R(x)]\nabla H = g^{\perp}(x)[J_d(x) - R_d(x)]\nabla H_d, \tag{5.17}$$

where $g^{\perp}(x)$ is a full–rank left annihilator of $g(x)$, i.e., $g^{\perp}(x)g(x) = 0$, and $H_d(x)$ is such that

$$x_\star = \arg\min H_d(x), \tag{5.18}$$

with $x_\star \in \mathbb{R}^n$ the equilibrium to be stabilized. Then, the closed–loop system (5.1) with $u = \beta(x)$, where

$$\beta(x) = [g^T(x)g(x)]^{-1}g^T(x)\{[J_d(x) - R_d(x)]\nabla H_d - [J(x) - R(x)]\nabla H\}, \tag{5.19}$$

takes the port-Hamiltonian form

$$\dot{x} = [J_d(x) - R_d(x)]\nabla H_d, \tag{5.20}$$

with $x_\star$ a (locally) stable *equilibrium. It will be* asymptotically *stable if, in addition, $x_\star$ is an isolated minimum of $H_d(x)$ and the largest invariant set under the closed–loop dynamics (5.20) contained in*

$$\left\{x \in \mathbb{R}^n \mid [\nabla H_d]^T R_d(x) \nabla H_d = 0\right\}, \tag{5.21}$$

equals $\{x_\star\}$. An estimate of its domain of attraction *is given by the largest* bounded *level set $\{x \in \mathbb{R}^n \mid H_d(x) \leq c\}$.*

Following the ideas of Chapter 2 and 3 the IDA–PBC methodology can be expressed in the Dirac structure framework as follows: consider the port–Hamiltonian system with state space $\mathcal{X}$, Hamiltonian H corresponding to the energy storage port $\mathcal{S}$, resistive port $\mathcal{R}$ and control port $\mathcal{C}$, given in input–state–output form in (5.1). If the Dirac structure $\mathcal{D}$ is given in matrix kernel representation as

$$\begin{aligned}\mathcal{D} = \{&(f_S, e_S, f_R, e_R, f_c, e_c) \in \mathcal{F}_S \times \mathcal{F}_S^* \times \mathcal{F}_R \times \mathcal{F}_R^* \times \mathcal{F}_c \times \mathcal{F}_c^* \\ &\mid F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_c f_c + E_c e_c = 0\},\end{aligned}$$

with

i) $E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E_c F_c^T + F_c E_c^T = 0,$

ii) $\mathrm{rank}[F_S \vdots E_S \vdots F_R \vdots E_R \vdots F_c \vdots E_c] = \dim(\mathcal{F}_R \times \mathcal{F}_R \times \mathcal{F}_c).$

Then, the port–Hamiltonian system (5.1) is given by the set of equations

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) + F_R f_R(t) + E_R e_R(t) + F_c f(t) + E_c e_c(t) = 0, \tag{5.22}$$

where we have set the flows of the energy storing elements $f_S = -\dot{x}$ (the negative sign is included to have a consistent energy flow direction) and the efforts corresponding to the energy storing elements $e_S = \frac{\partial H}{\partial x}$.

Restricting to linear resistive elements, the flow and effort variables connected to the resistive elements are related as $f_R = -\tilde{R} e_R$, with $\tilde{R} = \tilde{R}^T \geq 0$. Substituting these into (5.22) leads to the description of the physical system (5.1) by the set of DAE's

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) - F_R \tilde{R} e_R + E_R e_R + F_c f_c(t) + E_c e_c(t) = 0, \tag{5.23}$$

where we can see that (5.1) is a special case of (5.23) by letting

$$F_S = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}, E_S = \begin{bmatrix} J(x) \\ -g_R^T(x) \\ -g^T(x) \end{bmatrix}, F_R = \begin{bmatrix} g_R(x) \\ 0 \\ 0 \end{bmatrix}, E_R = \begin{bmatrix} 0 \\ I_r \\ 0 \end{bmatrix},$$
$$F_c = \begin{bmatrix} g(x) \\ 0 \\ 0 \end{bmatrix}, E_c = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix},$$

with $r = \dim F_R$, and setting $u = f_c, y = e_c$ and $R(x) = g_R^T(x)\tilde{R}g_R(x)$ with g_R representing the input matrix corresponding to the resistive port. As above the objective of IDA–PBC is to find a control input $u = \beta(x)$ such that the closed-loop system (5.20) in implicit form is given by

$$-F_S\dot{x}(t) + E_{S_d}\frac{\partial H_d}{\partial x}(x(t)) - F_{R_d}\tilde{R}_d e_{R_d} + E_{R_d}e_{R_d} = 0, \tag{5.24}$$

with

$$F_S = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}, E_{S_d} = \begin{bmatrix} J_d(x) \\ -g_{R_d}(x) \\ 0 \end{bmatrix}, F_{R_d} = \begin{bmatrix} g_{R_d}(x) \\ 0 \\ 0 \end{bmatrix}, E_{R_d} = \begin{bmatrix} 0 \\ I_{r_d} \\ 0 \end{bmatrix},$$

where $J_d(x) = -J_d^T(x)$, $R_d(x) = g_{R_d}^T(x)\tilde{R}_d g_{R_d}(x) = R_d^T(x) \geq 0$, with g_{R_d} representing the input matrix corresponding to the desired resistive port and $r_d = \dim F_{R_d}$.

Multiplying both sides of (5.23) and (5.24) by $F_c^\perp$—a full–rank left annihilator of F_c, i.e. $F_c^\perp F_c = 0$—and eliminating $\dot{x}$, we get

$$\begin{aligned} &F_c^\perp\left[E_S\frac{\partial H}{\partial x}(x(t)) - F_R\tilde{R}e_R + E_Re_R + E_ce_c(t)\right] \\ &= F_c^\perp\left[E_{S_d}\frac{\partial H_d}{\partial x}(x(t)) - F_{R_d}\tilde{R}_d e_{R_d} + E_{R_d}e_{R_d}\right], \end{aligned}$$

assigning $F_c^\perp = [g^\perp(x) \quad 0 \quad 0]$, with $g^\perp(x)$ a full–rank left annihilator of $g(x)$, that is, $g^\perp(x)g(x) = 0$, the above equation becomes

$$F_c^\perp\left[E_S\frac{\partial H}{\partial x}(x(t)) - F_R\tilde{R}e_R\right] = F_c^\perp\left[E_{S_d}\frac{\partial H_d}{\partial x}(x(t)) - F_{R_d}\tilde{R}_d e_{R_d}\right], \tag{5.25}$$

which is an equivalent representation of the *matching condition* (5.17).

Remark 5.12. Equation (5.25) gives us the equivalent matching condition of the input state output port-Hamiltonian system (5.1) written in a matrix-kernel representation. This analysis could give way for developing IDA-PBC techniques for general port-Hamiltonian systems not necessarily in the input-output form.

5.2 Control of infinite-dimensional systems

5.2.1 Stability of infinite-dimensional systems

In contrast to the finite-dimensional system case the stability analysis of infinite-dimensional systems is more complicated, even though the idea is again to show that the equilibrium corresponds to a strict extremum of the total energy. However, to establish stability it is no longer sufficient to examine the definiteness of the second variation of the Lyapunov function. In infinite-dimensions care must be taken to specify the norm associated with the stability argument because stability with respect to one norm does not necessarily imply stability with respect to another norm. This is a consequence of the fact that, unlike finite-dimensional vector spaces, all norms are not equivalent in infinite dimensions. In particular, in infinite-dimensions, not every convergent sequence in the unit ball converges to a point on the unit ball, that is unit balls in infinite-dimensional spaces need not be compact.

Definition 5.13. The equilibrium point χ_{r*} of a distributed parameter system is said to be stable in the sense of Lyapunov with respect to the norm $\| \cdot \|$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\| \chi_r(0) - \chi_{r*} \| < \delta \implies \| \chi_r - \chi_{r*} \| < \epsilon$ for all $t > 0$, where $\chi_r(0)$ is the initial condition of χ_r.

We state a stability theorem for infinite-dimensional systems, what is referred to as Arnold's theorem for stability of infinite-dimensional systems.

Theorem 5.14. *(Stability of an infinite-dimensional system): Consider a dynamical system $\dot{x} = f(x)$ on a linear space $\mathcal{X}$, where x_* is an equilibrium. Assume there exist solutions to the system and suppose there exists a function $H_d : \mathcal{X} \to \mathbb{R}$ such that*

$$\delta H_d(x_*) = 0 \quad \textit{and} \quad \frac{dH_d}{dt} \leq 0.$$

Define $\Delta x = x - x_*$ and assume there exists a quadratic function Q such that

$$c_1 Q(\Delta x) \leq H_d(x_* + \Delta x) - H_d(x_*), \tag{5.26}$$

while $Q(\Delta x) > 0$ for all $\Delta x \neq 0$. Define the norm $\| \Delta x \|$ by $\| \Delta x \|^2 = Q(\Delta x)$. Assume that

$$| H_d(x_* + \Delta x) - H_d(x_*) | \leq c_2 \| \Delta x \|^\alpha, \tag{5.27}$$

for certain constants $\alpha, c_1, c_2 > 0$ and $\| \Delta x \|$ sufficiently small. Then x_* is a stable equilibrium.

Proof. Since $H_d(x + x_*)$ is decreasing in time, we obtain from (5.26) for all $t \geq 0$

$$\| \Delta x \|^2_{time=t} \leq \| H_d(x_* + \Delta x) - H_d(x_*) \|_{time=t} \leq \| H_d(x_* + \Delta x) - H_d(x_*) \|_{time=0} \ .$$

Hence by (5.27)

$$\| \Delta x \|^2_{time=t} \leq \frac{c_2}{c_1} \| \Delta x \|^{\alpha}_{time=0},$$

showing stability. □

5.2.2 Control by Interconnection: example of an RLC circuit with a transmission line

As an example of the above result on stability of infinite-dimensional system, we consider the case of stabilization of a plant system which is a composition of an infinite-dimensional port-Hamiltonian system with a finite dimensional system. This is a case of what we called as mixed finite and infinite-dimensional port-Hamiltonian systems in Chapter 3. We consider stabilization of a RLC circuit with a transmission line.

Consider a series RLC circuit whose dynamics are given by the following set of equations:

$$\begin{bmatrix} f_x \\ e \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & R \end{bmatrix} & -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} e_x \\ f \end{bmatrix}, \tag{5.28}$$

where $(f_x, f, e_x, e) = (-[\dot{x}_1, \dot{x}_2]^T, u_p, \left[\frac{x_1}{C}, \frac{x_2}{L}\right]^T, y_p)$.

The total energy of the circuit is $H(x) = \frac{1}{2}\frac{x_1^2}{C} + \frac{1}{2}\frac{x_2^2}{L}$, where $x_1 = q$, the charge on the capacitor and $x_2 = \phi$ the flux in the inductor. The dynamics of the transmission line are given by the telegraphers equations

$$\begin{bmatrix} \frac{\partial}{\partial t} q \\ \frac{\partial}{\partial t} \phi \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ -\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H} \\ \delta_\phi \mathcal{H} \end{bmatrix}, \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H} \mid_b \\ \delta_\phi \mathcal{H} \mid_b \end{bmatrix}. \tag{5.29}$$

We consider a port-Hamiltonian plant whose dynamics are described by (5.28) interconnected to a port-Hamiltonian controller system through a transmission line (an infinite dimensional system) given by (5.29). The interconnection constraints are of the form

$$\begin{aligned} y_c &= f_0, \quad u_c = e_0, \\ y_p &= e_l, \quad u_p = -f_l. \end{aligned} \tag{5.30}$$

With the above interconnection constraints the closed-loop dynamics can be written as

$$
\begin{aligned}
\begin{bmatrix} f_p \\ f_c \\ f_q \\ f_\phi \end{bmatrix} &= \begin{bmatrix} -[J(x)-R(x)] & 0 & 0 & 0 \\ 0 & -[J_c(\xi)-R_c(\xi)] & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d} \\ 0 & 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_c \\ e_q \\ e_\phi \end{bmatrix} + \\
&\quad \begin{bmatrix} -g(x) & 0 \\ 0 & -g_c(\xi) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_{\phi l} \\ e_{q0} \end{bmatrix} \\
\begin{bmatrix} e_{ql} \\ e_{\phi 0} \end{bmatrix} &= \begin{bmatrix} g^T(x) e_p \\ -g_c(\xi) e_c \end{bmatrix}.
\end{aligned}
\tag{5.31}
$$

In energy variables the overall dynamics is given as

$$
\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{\xi} \\ \frac{\partial}{\partial t} q(z,t) \\ \frac{\partial}{\partial t} \phi(z,t) \end{bmatrix} &= \begin{bmatrix} [J(x)-R(x)] & 0 & 0 & 0 \\ 0 & [J_c(\xi)-R_c(\xi)] & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d} \\ 0 & 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} H(x) \\ \frac{\partial}{\partial \xi} H(\xi) \\ \delta_p \mathcal{H}(\bar{q}) \\ \delta_\phi \mathcal{H}(\bar{q}) \end{bmatrix} + \\
&\quad \begin{bmatrix} g(x) & 0 \\ 0 & g_c(\xi) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_\phi \mathcal{H}(\bar{q}) \mid_l \\ \delta_q \mathcal{H}(\bar{q}) \mid_0 \end{bmatrix} \\
\begin{bmatrix} \delta_q \mathcal{H}(\bar{q}) \mid_l \\ \delta_\phi \mathcal{H}(\bar{q}) \mid_0 \end{bmatrix} &= \begin{bmatrix} g^T(x) \frac{\partial}{\partial x} H(x) \\ -g_c(\xi) \frac{\partial}{\partial \xi} H(\xi) \end{bmatrix}.
\end{aligned}
$$

The closed-loop energy defined in the extended state space $\chi = [x, \xi, q(z,t), \phi(z,t)]^T$ is given by

$$
H_{cl}(\chi) = H(x) + H_c(\xi) + \mathcal{H}(\bar{q}),
$$

with energy rate

$$
\dot{H}_{cl} = -\frac{\partial^T H}{\partial x}(x) R(x) \frac{\partial^T H}{\partial x}(x) - \frac{\partial^T H_c}{\partial \xi}(\xi) R_c(\xi) \frac{\partial H_c}{\partial \xi}(\xi).
$$

A function $C(\chi)$ will be a Casimir function provided (this is a consequence of

Equation (4.54))

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}-[J(x)-R(x)] & 0 & 0 & 0\\ 0 & -[J_c(\xi)-R_c(\xi)] & 0 & 0\\ 0 & 0 & 0 & \mathrm{d}\\ 0 & 0 & \mathrm{d} & 0\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}C(\chi)\\ \frac{\partial}{\partial \xi}C(\chi)\\ \delta_p C(\chi)\\ \delta_\phi C(\chi)\end{bmatrix} + \begin{bmatrix}-g(x) & 0\\ 0 & -g_c(\xi)\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_\phi C(\chi)|_l\\ \delta_q C(\chi)|_0\end{bmatrix}$$

$$\begin{bmatrix}\delta_q C(\chi)|_l\\ \delta_\phi C(\chi)|_0\end{bmatrix} = \begin{bmatrix}g^T(x)\frac{\partial}{\partial x}C(\chi)\\ -g_c(\xi)\frac{\partial}{\partial \xi}C(\chi)\end{bmatrix}. \tag{5.32}$$

The third and fourth relation of (5.32) says that every Casimir function should be linear with respect to the spatial variables i.e.

$$\delta_\phi C(\chi),\ \delta_q C(\chi) = \text{constant as a function of } z, \tag{5.33}$$

(this is also consistent with the expression we obtained in Equation 4.47). Hence C should satisfy

$$\delta_\phi C(\chi) = \delta_\phi C(\chi)\ |_0 = \delta_\phi C(\chi)\ |_l = -g(\xi)\frac{\partial}{\partial \xi}C(\chi)$$
$$\delta_q C(\chi) = \delta_q C(\chi)\ |_0 = \delta_q C(\chi)\ |_l = g(\xi)\frac{\partial}{\partial x}C(\chi),$$

with the above equations (5.32) reduce to

$$\begin{bmatrix}J(x)-R(x) & -g(x)g_c^T(\xi)\\ g_c(\xi)g^T(x) & J_c(\xi)-R_c(\xi)\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}C(\chi)\\ \frac{\partial}{\partial \xi}C(\chi)\end{bmatrix} = 0$$

$$\begin{bmatrix}\delta_q C(\chi)\\ \delta\phi C(\chi)\end{bmatrix} = \begin{bmatrix}g^T(x)\frac{\partial}{\partial x}C(\chi)\\ -g(\xi)\frac{\partial}{\partial \xi}C(\chi)\end{bmatrix}.$$

We consider Casimirs of the form

$$C(\chi) = F(x,\xi) + \mathcal{F}(\bar{q}(z,t)). \tag{5.34}$$

which means that we are looking for functions which satisfy

$$\begin{bmatrix}J(x)-R(x) & -g(x)g_c^T(\xi)\\ g_c(\xi)g^T(x) & J_c(\xi)-R_c(\xi)\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}F(x,\xi)\\ \frac{\partial}{\partial \xi}F(x,\xi)\end{bmatrix} = 0$$
$$\begin{bmatrix}\delta_q \mathcal{F}(\bar{q}((z,t))\\ \delta_\phi \mathcal{F}(\bar{q}(z,t))\end{bmatrix} = \begin{bmatrix}g^T(x)\frac{\partial}{\partial x}F(x,\xi)\\ -g(\xi)\frac{\partial}{\partial \xi}F(x,\xi)\end{bmatrix}. \tag{5.35}$$

The first conditions of (5.35) are the same as those for a finite dimensional plant-controller interconnection:

$$
\begin{aligned}
\frac{\partial^T F}{\partial x}(x,\xi)J(x)\frac{\partial F}{\partial x} &= \frac{\partial^T F}{\partial \xi}(x,\xi)J_c(\xi)\frac{\partial F}{\partial \xi} \\
R(x)\frac{\partial F}{\partial x}(x,\xi) &= R_c(\xi)\frac{\partial F}{\partial \xi} = 0 \\
\frac{\partial^T F}{\partial x}(x,\xi)J(x) &= -\frac{\partial^T F}{\partial \xi}(x,\xi)g_c(\xi)g^T(x) \\
\frac{\partial^T C}{\partial \xi}(x,\xi)J_c(\xi) &= \frac{\partial^T F}{\partial x}(x,\xi)g(x)g_c^T(\xi),
\end{aligned}
\tag{5.36}
$$

and the conditions on the functional are

$$
\begin{aligned}
\delta_q \mathcal{F}(\bar{q}(z,t)) &= g^T(x)\frac{\partial F}{\partial x}(x,\xi) \\
\delta_\phi \mathcal{F}(\bar{q}(z,t)) &= -g_c^T(\xi)\frac{\partial F}{\partial \xi}(x,\xi).
\end{aligned}
\tag{5.37}
$$

Thus we have proved the following

Proposition 5.15. *The functions $C(\chi) = F(x,\xi) + \mathcal{F}(\bar{q}(z,t))$ are Casimir functions of the interconnected port-Hamiltonian system (5.31) if and only if the function $F(x)$ satisfies (5.36) and the functional $F(\bar{q}(z,t))$ satisfies (5.37) and (5.33) if $y_p = e_l$ or*

$$
\begin{aligned}
\delta_q \mathcal{F}(\bar{q}(z,t)) &= -g_c^T(\xi)\frac{\partial F}{\partial x}(x,\xi) \\
\delta_\phi \mathcal{F}(\bar{q}(z,t)) &= g^T(x)\frac{\partial F}{\partial x}(x,\xi),
\end{aligned}
$$

if $y_p = e_l$.

Control Design

In the case of mixed lumped and distributed parameter systems we will define stability in the sense of Lyapunov as follows:

A suitable norm in our case would be

$$
\| \Delta\chi_r \| = \left(|\Delta x|^2 + |\Delta\xi|^2 + \int_0^l \Delta\tilde{q}^2(z,t)dz + \int_0^l \Delta\tilde{\phi}^2(z,t)dz \right)^{\frac{1}{2}}, \tag{5.38}
$$

with $|\cdot|$ the standard Euclidean norm and $q = \tilde{q}dz$ and $\phi = \tilde{\phi}dz$.

We propose a controller of the following form:

$$\dot{\xi} = u_c$$

$$y_c = \tfrac{\partial H_c}{\partial \xi}(\xi),$$

and have the closed-loop system as the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi} \\ \frac{\partial}{\partial t} q(z,t) \\ \frac{\partial}{\partial t}\phi(z,t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{d} \\ 0 & 0 & 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{x_1}{C} \\ \frac{x_2}{L} \\ \frac{\partial H_c}{\partial \xi}(\xi) \\ \frac{*q(z,t)}{C(z)} \\ \frac{*\phi(z,t)}{L(z)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{*q(l,t)}{C(l)} \\ \frac{*\phi(0,t)}{L(0)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{*\lambda(l,t)}{L(l)} \\ \frac{*q(0,t)}{C(0)} \end{bmatrix} = \begin{bmatrix} \frac{x_2}{L} \\ -\frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}.$$

The total closed loop energy function is defined by

$$H_{cl}(\chi) = \frac{1}{2}\frac{x_1^2}{C} + \frac{1}{2}\frac{x_2^2}{L} + H_c(\xi) + \frac{1}{2}\int_0^l \left(\frac{\tilde{q}^2(z,t)}{C(z)} + \frac{\phi^2(\tilde{z},t)}{L(z)}\right) dz,$$

with

$$\dot{H}_{cl}(\chi) = -R\left(\frac{x_2^2}{L}\right)^2.$$

Then the conditions on the Casimirs are as follows:

$$\begin{aligned} &\frac{\partial^T F}{\partial x}(x,\xi) J(x) \frac{\partial F}{\partial x} = 0 \\ &R(x)\frac{\partial F}{\partial x}(x,\xi) = 0 = R_c(\xi) \\ &\frac{\partial^T F}{\partial x}(x,\xi) J(x) = -\frac{\partial^T F}{\partial \xi}(x,\xi) g^T(x) \\ &\frac{\partial^T F}{\partial x}(x,\xi) g^T(x) = 0, \end{aligned}$$

and the conditions on the functional are

$$\begin{aligned} &\delta_\phi \mathcal{F}(\bar{q}(z,t)) = 0 \\ &\delta_q \mathcal{F}(\bar{q}(z,t)) = -\frac{\partial F}{\partial \xi}(x,\xi). \end{aligned}$$

Hence the function $F(x, \xi)$ should satisfy the following PDE

$$\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial \xi} = 0. \tag{5.39}$$

It is clear that any function of the form $f(x_1 - \xi)$ satisfies the above equation, and the functional $\mathcal{F}(\bar{q}(z,t))$ should be of the following form,

$$\mathcal{F}(\bar{q}(z,t)) = -\frac{\partial F}{\partial \xi}(x, \xi) \int_0^l \tilde{q}(z,t)dz. \tag{5.40}$$

Hence the Casimir function in the extended state space is given by

$$\mathcal{C}(\chi) = f(x_1 - \xi) - \frac{\partial F}{\partial \xi}(x, \xi) \int_0^l \tilde{q}(z,t)dz, \tag{5.41}$$

and the Lyapunov function is given by

$$\begin{aligned} H_d(\chi_r) = & \; \tfrac{1}{2}\tfrac{x_1^2}{C} + \tfrac{1}{2}\tfrac{x_2^2}{L} + H_c(\xi) + \\ & \tfrac{1}{2}\int_0^l (\tfrac{\tilde{q}^2(z,t)}{C(z)} + \tfrac{\tilde{\phi}^2(z,t)}{L(z)})dz + f(x_1 - \xi) - \tfrac{\partial F}{\partial \xi}(x, \xi) \int_0^l \tilde{q}(z,t)dz. \end{aligned} \tag{5.42}$$

Next we show that by selecting

$$H_c(\xi) = k_1 \tilde{\xi} + k_2 \tilde{\xi}^2, \tag{5.43}$$

and the function $f(x_1 - \xi)$ as

$$f(x_1 - \xi) = k_1(\tilde{x}_1 - \tilde{\xi}),$$

where $(\tilde{x}_1, \tilde{\xi}) = (x_1 - x_1^*, \xi - \xi_*)$, the shifted equilibrium, with ξ_*, the controller equilibrium which can be chosen arbitrarily. We can then shape the total energy in such a way that it has a minimum at the equilibrium point $\chi_{r*} = [x_*, \xi_*, q_*(z), \lambda_*(z)]^T$,

$$\nabla H_d(\chi_{r*}) = \begin{bmatrix} \frac{x_{1*}}{C} + k_1 \\ 0 \\ 0 \\ \frac{\tilde{q}_*(z)}{C(z)} + k_1 \\ \frac{\tilde{\lambda}_*(z)}{L(z)} \end{bmatrix} = 0 \implies \begin{bmatrix} k_1 = -\frac{x_{1*}}{C} \\ \tilde{q}_*(z) = C(z)\frac{x_{1*}}{C} \\ \tilde{\lambda}_*(z) = 0 \end{bmatrix}. \tag{5.44}$$

Then we verify that these first order conditions are compatible with the boundary conditions imposed by the interconnection constraints at the equilibrium point, which are given by

$$y_{p*} = -\frac{\tilde{\phi}_*(l)}{L(l)} = 0, \qquad u_{c*} = -\frac{\tilde{\phi}_*(0)}{L(0)},$$

$$u_{p*} = \frac{\tilde{q}_*(l)}{C(l)} = -\frac{x_{1*}}{C}, \quad y_{c*} = -\frac{\tilde{q}_*(0)}{C(0)},$$

and using the first order conditions and the fact that $y_{c*} = \frac{\partial H_c}{\partial \xi}(\xi_*)$ we have

$$\tilde{\phi}_*(l) = 0 \qquad\qquad u_{c*} = 0$$

$$\tilde{q}_*(l) = C(l)\tfrac{x_{1*}}{C} \quad \tilde{q}_*(0) = -k_1 C(0),$$

and hence the boundary conditions imposed by the equilibrium point and the first order condition are compatible.

In order to verify the second order conditions, we compute the functional $H_d(\chi_* + \Delta\chi)$ in (5.26) as proportional to the second variation of $H_d(\chi_r)$ in the sense that its Taylor expansion about $\Delta\chi_r$ is

$$\begin{aligned}\mathcal{N}(\Delta\chi_r) &= H_d(\chi_{r*} + \Delta\chi_r) - H_d(\chi_{r*}) \\ &\approx \frac{1}{2}\nabla^2 H_d(\chi_{r*}).\end{aligned} \tag{5.45}$$

We then get

$$\begin{aligned} H_d(\chi_{r*} + \Delta\chi_r) = \;& \tfrac{1}{2}\tfrac{(\frac{x_{1*}}{C}+\Delta x_1)^2}{C} + \tfrac{1}{2}\tfrac{(\Delta x_2)^2}{L} + k_1(\Delta\xi) + k_2(\Delta\xi)^2 + \\ & \tfrac{1}{2}\int_0^l \left(\tfrac{(C(z)\frac{x_{1*}}{C}+\Delta q(z,t))^2}{C(z)} + \tfrac{(\Delta\phi(z,t))^2}{L(z)}\right) dz + k_1(\Delta x_1 - \Delta\xi) + \\ & k_1 \int_0^l (C(z)\tfrac{x_{1*}}{C} + \Delta\tilde{q}(z,t))dz, \end{aligned}$$

and

$$H_d(\chi_{r*}) = \frac{1}{2}\frac{(\frac{x_{1*}}{C})^2}{C} + \frac{1}{2}\int_0^l (\frac{(C(z)\frac{x_{1*}}{C})^2}{C(z)} + k_1\int_0^l C(z)\frac{x_1*}{C}dz,$$

where we have used (5.44) therefore,

$$\mathcal{N}(\Delta\chi_r) = \frac{1}{2}\frac{\Delta x_1^2}{C} + \frac{1}{2}\frac{\Delta x_2^2}{L} + k_2(\Delta\xi)^2 + \frac{1}{2}\int_0^l \left(\frac{\Delta q^2(z,t)}{C(z)} + \frac{\Delta\phi^2(z,t)}{L(z)}\right) dz. \tag{5.46}$$

Now we verify conditions (5.26) and (5.27) with respect to the following norm.

$$\| \chi_r \| = \left(\Delta x_1^2 + \Delta x_2^2 + \Delta\xi^2 + \int_0^l \Delta\tilde{q}^2(z,t)dz + \int_0^l \Delta\tilde{\phi}^2(z,t)dz\right)^{\frac{1}{2}}.$$

Considering that the physical characteristics of the transmission line (capacitance and inductance) are upper and lower bounded on $[0, l]$, that is

$$L_m \leq \frac{1}{L(z)} \leq L_M,\; C_m \leq \frac{1}{C(z)} \leq C_M,\; L_i, C_i > 0,\; i = M, m. \tag{5.47}$$

It is easy to see that we can find constants $c_{11}, c_{12}, c_{q1}, c_{q2}$ which satisfy

$$c_{11}\Delta x_1^2 \leq \frac{1}{2}\frac{\Delta x_1^2}{C} \leq c_{12}\Delta x_1^2$$

$$c_{q1}\int_0^l \Delta q^2(z,t)dz \leq \int_0^l \frac{\Delta q^2(z,t)}{C(z)}dz \leq c_{q2}\int_0^l \Delta q^2(z,t)dz,$$

and constants $c_{21}, c_{22}, c_{\lambda 1}, c_{\lambda 2}$ which satisfy

$$c_{21}\Delta x_2^2 \leq \frac{1}{2}\frac{\Delta x_2^2}{L} \leq c_{22}\Delta x_2^2$$

$$c_{\lambda 1}\int_0^l \Delta \phi^2(z,t)dz \leq \int_0^l \frac{\Delta \phi^2(z,t)}{L(z)}dz \leq c_{\lambda 2}\int_0^l \Delta \phi^2(z,t)dz,$$

and finally constants $c_{31,}c_{32}$ which satisfy

$$c_{31}\Delta\xi^2 \leq k_2(\Delta\xi)^2 \leq c_{32}\Delta\xi^2.$$

In fact one can simply take the constants c_{11}, c_{12} and c_{21}, c_{22} as

$$c_{11} = c_{12} = \frac{1}{2C}$$

$$c_{21} = c_{22} = \frac{1}{2L}.$$

Finally we have

$$c_1 \triangleq \min\{\frac{1}{2C}, \frac{1}{2L}, c_{31}, c_{q1}, c_{\lambda 1}\}$$

$$c_2 \triangleq \max\{\frac{1}{2C}, \frac{1}{2L}, c_{32}, c_{q2}, c_{\lambda 2}\}.$$

Hence we have proved

Proposition 5.16. *Consider the RLC circuit defined by (5.28), the transmission line modeled by (5.29), and the port-Hamiltonian controller defined by*

$$\dot{\xi} = -\frac{\tilde{\phi}(0,t)}{L(0)}$$
$$y_c = -\frac{x_{1*}}{C} + 2k_2(\xi - \xi_*),$$

under the interconnection constraints (5.30). The resulting interconnected system has a stable equilibrium in the sense of definition (5.13) at

$$\chi_* = \left[\frac{x_{1*}}{C}, 0, \xi_*, C(z)\frac{x_{1*}}{C}, 0\right]^T.$$

5.2.3 The La Salle's principle approach

La Salle's theorem is a well-known result for the stability analysis of finite-dimensional non-linear systems. If in a domain about the equilibrium point we can find a Lyapunov function $V(x)$ whose derivative along the trajectories of the system is only negative semi-definite and if we can establish that no trajectories, other than the equilibrium, can stay in the region where $\dot{V}(x) = 0$, then this configuration is asymptotically stable. This is referred to as the La Salle's invariance principle.

To generalize this to infinite-dimensional systems, consider an infinite-dimensional port-Hamiltonian system given by (2.52) and denote by $\mathcal{X}_\infty$ its infinite-dimensional state space. Then assuming existence of solutions, it is possible to define an operator $\Phi(t) : \mathcal{X}_\infty \to \mathcal{X}_\infty$ such that

$$(\alpha_p, \alpha_q)(t) = \Phi(t)(\alpha_p, \alpha_q)(0),$$

for each $t \geq 0$. It can then be proven that $\Phi(t)$ is a family of bounded and continuous operators which is called C_0 semi-group on $\mathcal{X}$. See [34] for details. The operator Φ gives the solution of the infinite-dimensional port-Hamiltonian system (2.52) for given initial and boundary conditions. For every χ, denote by

$$\gamma(\chi) := \bigcup_{t \geq 0} \Phi(t)\chi, \tag{5.48}$$

the set of all orbits of the infinite-dimensional port-Hamiltonian system through χ and by

$$\omega(\chi) := \left\{ \bar{\chi} \in \mathcal{X}_\infty \mid \bar{\chi} = \lim_{n \to \infty} \Phi(t_n)\chi, \text{ with } t_n \to \infty \text{ as } n \to \infty \right\},$$

the (possibly empty) ω-limit set of χ. It can be shown [23] that $\omega(\chi)$ is always positively invariant, i.e. $\Phi(t)\omega(\chi) \subset \omega(\chi)$. Moreover, $\omega(\chi)$ is also closed.

Theorem 5.17. *[23] If $\chi \in \mathcal{X}_\infty$ and $\gamma(\chi)$ is precompact [2], then $\omega(\chi)$ is nonempty, compact and connected. Moreover,*

$$\lim_{t \to \infty} d(\Phi(t)\chi, \omega(\chi)) = 0,$$

where, given $\bar{\chi} \in \mathcal{X}_\infty$ and $\Omega \subset \mathcal{X}_\infty$, $d(\bar{\chi}, \Omega)$ denotes the distance from $\bar{\chi}$ to Ω, that is

$$d(\bar{\chi}, \Omega) = \inf_{\omega \in \Omega} ||\bar{\chi} - \omega||.$$

[2] A relatively compact subset Y of a topological space X is a subset whose closure is compact. Since closed subsets of compact spaces are compact, every set in a compact space is relatively compact. In the case of a metric topology, or more generally when sequences may be used to test for compactness, the criterion for relative compactness becomes that any sequence in Y has a subsequence convergent in X. This condition is also called pre-compact or relatively bounded.

This theorem characterizes the asymptotic behavior of the distributed parameter systems once the ω-limit set is calculated. Based on this result, it is possible to state the invariance principle.

Theorem 5.18. *[23] (LaSalle' s invariance principle) Denote by $\mathcal{H}_\infty$ a Lyapunov function for the system (2.52), that is for $\Phi(t)$, and by $\mathcal{B}$ the largest invariant subset of*

$$\left\{\chi \in \mathcal{X}_\infty \mid \dot{\mathcal{H}}_\infty(\chi) = 0\right\},$$

that is $\Phi(t)\mathcal{B} = \mathcal{B}$ for all $t \geq 0$. If $\chi \in \mathcal{X}_\infty$ and $\gamma(\chi)$ is precompact, then

$$\lim_{t\to\infty} d(\Phi(t)\chi, \mathcal{B}) = 0.$$

An immediate consequence is expressed by the following corollary.

Corollary 5.19. *Consider an infinite-dimensional port-Hamiltonian system for which we assume existence of solutions and that $\gamma(\chi)$ defined in (5.48) is precompact. Denote by χ_* an equilibrium point and by $\mathcal{H}_\infty$ a Lyapunov function. If the largest invariant subset of*

$$\left\{\chi \in \mathcal{X}_\infty \mid \dot{\mathcal{H}}_\infty(\chi) = 0\right\}$$

equals $\{\chi_*\}$, then χ_* is asymptotically stable.

5.2.4 Control by damping injection

In the case of finite-dimensional systems, we know that if the energy function $\mathcal{H}$ of the system is characterized by a minimum at χ^*, then it is possible to drive the system to the desired configuration by interconnecting a controller that behaves as a dissipative element to the plant.

Now consider an infinite-dimensional port-Hamiltonian systems with port-variables $(f_p, f_q, f_b, e_p, e_q, e_b)$ corresponding to the variables in the spatial domain Z and also the boundary ∂Z. We focus on control through the boundary of the system, more precisely control by damping injection through the boundary. Consider the map

$$S : \Omega^{n-q}(\partial Z) \to \Omega^{n-p}(\partial Z),$$

satisfying

$$S(e_b) \wedge e_b \geq 0. \tag{5.49}$$

We say that boundary damping is introduced if we relate the flows and effort variables at the boundary by

$$f_b = -S(e_b). \tag{5.50}$$

From the energy balance equation (2.53) we have

$$\frac{d\mathcal{H}}{dt} = -\int_{\partial Z} S(e_b) \wedge e_b \leq 0.$$

Consequently, the energy function is non-increasing along the system trajectories and it reaches a steady state configuration when

$$S(e_b) \wedge e_b = 0, \tag{5.51}$$

on the boundary, where dissipation enters into the system. Now, denote by $\mathcal{B}$ the set of configuration χ compatible with (5.51). We can then state the following proposition, the proof of which follows from the La Salle's principle.

Proposition 5.20. *Consider the infinite-dimensional port-Hamiltonian for which solutions exist and which satisfy the precompactness conditions. Consider the boundary control (5.50). If the largest invariant subset of*

$$\left\{\chi \mid \dot{\mathcal{H}}(\chi) = 0\right\} \cap \mathcal{B},$$

equals χ^*, then the configuration χ^* is asymptotically stable.

Example 5.21 (Example 3.3.2 continued). Consider the coupled wave equations of (3.36). The total energy of the system is given by

$$\begin{aligned} H &= \frac{1}{2}\int_Z \left[(\epsilon_1 \wedge \sigma_1 + \rho_1 \wedge v_1) + (\epsilon_2 \wedge \sigma_2 + \rho_2 \wedge v_2) + kq^2\right] \\ &= \frac{1}{2}\int_Z \left[(\epsilon_1 \wedge E_1 * \epsilon_1 + \rho_1 \wedge \frac{1}{\mu_1} * \rho_1) + (\epsilon_2 \wedge E_2 * \epsilon_{21} + \rho_2 \wedge \frac{1}{\mu_2} * \rho_2) + kq^2\right], \end{aligned}$$

with the energy rate given by

$$\frac{dH}{dt} = \left[\sigma_1.v_1 + \sigma_2.v_2\right]\Big|_0^l .$$

The control objective is to design a boundary control law under which the energy functional asymptotically assumes its zero configuration which is given by

$$\begin{aligned} &\epsilon_i = 0, \quad \rho_i = 0, \quad i = 1, 2 \\ &q = 0. \end{aligned} \tag{5.52}$$

If some dissipation effect is introduced through the boundary of the system then it is possible to drive the state of the vibrating strings, which are connected in parallel, to the configuration where the energy assumes its minimum, in other words the configuration where the vibrations in the strings are damped out, for all initial conditions.

Suppose we now interconnect the system with a controller at $z = L$, with the following conditions

$$v_i(0,t) = 0, \quad \sigma_i(L,t) = -\beta_i v_i(L,t), \quad t \geq 0, \quad i = 1, 2,$$

where $\beta_i > 0$ then we have

$$\frac{dH}{dt} = -\beta_1 v_1^2(L,t) - \beta_2 v_2^2(L,t) \leq 0.$$

If we know *a priori* that the trajectories are bounded, that is $\gamma(\chi)$ is precompact, then we know from proposition (5.20) that the system trajectories converge to the largest invariant set in (5.52) and hence we can say that the system asymptotically reaches its zero configuration. For techniques to compute pre-compactness of the set $\gamma(\chi)$, we refer to [23].

Remark 5.22. Similarly we can also look at other boundary control laws where we have dissipation in only one of the vibrating strings, these conditions would be given as

$$\begin{aligned} v_1(0,t) &= 0, \quad v_1(L,t) = 0 \\ v_2(0,t) &= 0, \quad \sigma_2(L,t) = -\beta_2 v_2(L,t), \end{aligned}$$

or

$$\begin{aligned} v_1(0,t) &= 0, \quad \sigma_2(L,t) = -\beta_1 v_1(L,t) \\ v_2(0,t) &= 0, \quad v_2(L,t) = 0. \end{aligned}$$

5.2.5 Energy based Lyapunov functions for infinite-dimensional systems

So far in the section of control of infinite-dimensional systems, we considered two different problems:

- In the first case we considered the problem of stabilization where the plant system consisted of an infinite-dimensional subsystem. The problem there was to stabilize a finite-dimensional system with a finite-dimensional controller via an infinite-dimensional system.
- Next we have seen how boundary damping enables a system to asymptotically get to its zero state.

We now use the techniques above to study stability of a "forced"infinite-dimensional system. The control, which is a constant input, enters through the boundary of the infinite-dimensional system

Consider an infinite-dimensional system, with a 1-D spatial domain and the Hamiltonian given by

$$\mathcal{H}(p,q) = \int_Z (c_1 \tilde{p}^2 + c_2 \tilde{q}^2) dz,$$

$p(z,t), q(z,t)$ are the state variables, with c_1 and c_2 which may be constants, as in the case of linear shallow water equations (2.64) or can depend on the spatial variable, for example the distributed capacitances and inductances in case of a transmission line (2.54). The dynamics of the system are given by

$$\begin{aligned}\partial_t\tilde{p}(z,t) + \partial_x(c_2\tilde{q}(z,t)) &= 0\\ \partial_t\tilde{q}(z,t) + \partial_x(c_1\tilde{p}(z,t)) &= 0,\end{aligned}$$

together with the boundary conditions given by the values of $c_1\tilde{p}$ and $c_2\tilde{q}$ (the variational derivatives of the Hamiltonian) evaluated at the boundary. The infinite-dimensional port-Hamiltonian system then satisfies the following energy balance equation

$$\begin{aligned}\frac{dH}{dt} &= f_l e_l - f_0 e_0\\ &= c_2\tilde{q}(l,t)c_1\tilde{p}(l,t) - c_2\tilde{q}(0,t)c_1\tilde{p}(0,t).\end{aligned} \tag{5.53}$$

The energy function has a minimum at $(p,q) = (0,0)$ and in most cases this equilibrium point is not the point of interest. Consider now a control problem where we desire to stabilize the system at a *forced* equilibrium point $(\bar{p},0)$, by making use of the boundary ports to *shape* the energy in such a way that it has a minimum at the desired equilibrium point. In most cases the Hamiltonian cannot be directly used for analyzing the stability of the forced equilibrium, since the right hand side of (5.53) will in general be non-positive. One way of approaching the problem is to start from the (5.53) and to being the second term on the right-hand side to the left hand side. The by adding damping though the boundary (through the first term on the right-hand side of (5.53)), we can look for Lyapunov function candidates of the form

$$H + \bar{f}_0\int_0^t e_0(\tau)d\tau, \tag{5.54}$$

where $\bar{f}_0$ is the value of the forced input. To check whether (5.54) can be used as a Lyapunov function, the question is if we can write $\bar{f}_0\int_0^t e_0(\tau)d\tau$ as a function of the state variable $p(z,t)$ (since we wish to shape the p component of the energy). From a system theoretic point of view, this suggests to consider interconnection of the constant source system to the plant system under consideration to one of its boundaries and look for Lyapunov functions of the interconnected system

$$V(\chi) = H(x) + H_s(\xi), \quad H_s(\xi) = -\bar{f}_0\xi,$$

with ξ being the state of the controller system. Writing $\bar{f}_0\int_0^t e_0(\tau)d\tau$ as a function of $p(z,t)$ then corresponds to expressing $\xi(t)$ as a function of $p(z,t)$ along the dynamics of the interconnected system.

For the sake of simple illustration, we consider the example of the transmission line whose dynamics are given by (2.54). We recall the equations here:

$$\begin{bmatrix} \partial_t q \\ \partial_t \phi \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_q H \\ \delta_\phi H \end{bmatrix} = \begin{bmatrix} \mathrm{d}\left(\frac{*q(z.t)}{C(z)}\right) \\ \mathrm{d}\left(\frac{*\phi(z,t)}{L(z)}\right) \end{bmatrix},$$

together with the boundary voltages and currents, the total energy in this case is given by

$$\mathcal{H}(q,\phi) = \frac{1}{2}\int_Z \left(\frac{\tilde{q}^2}{C} + \frac{\tilde{\phi}^2}{L}\right) dz.$$

In this case the problem would mean to stabilize the system at a desired voltage and zero current. Now consider the case where we have a constant input voltage at one end of the line. From a modeling perspective, we can write the source system as

$$\begin{aligned} \dot{\xi} &= u_s \\ y_s &= \frac{\partial H_s}{\partial \xi}, \end{aligned}$$

with H_s being the energy of the source system. We now interconnect this source system to the transmission line with the following interconnection constraints

$$\begin{aligned} e_\phi \mid_0 &= u_s \\ e_q \mid_0 &= -\frac{\partial H_s}{\partial \xi}. \end{aligned}$$

The resulting dynamics would be

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \partial_t q \\ \partial_t \phi \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \cdot \mid_0 \\ 0 & 0 & \mathrm{d} \\ 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_s}{\partial \xi} \\ \delta_q H \\ \delta_\phi H \end{bmatrix} \\ \delta_q H \mid_0 &= -\frac{\partial H_s}{\partial \xi} \\ f_l = \delta_q H \mid_l ; e_l &= \delta_\phi H \mid_l, \end{aligned}$$

with the total energy function

$$H_{cl}(q,p,\xi) = \frac{1}{2}\int \left(\frac{\tilde{q}^2}{C} + \frac{\tilde{\phi}^2}{L}\right) dz - \frac{\bar{q}}{C}\xi,$$

with $H_c(\xi) = -\frac{\bar{q}}{C}\xi$ being the energy (unbounded) of the source system. To this end we look for Casimirs of the form

$$F(q,p,\xi) = \mathcal{C}(p,q) - \xi,$$

by solving

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0 & 0 & \cdot\mid_0\\0 & 0 & \mathrm{d}\\0 & \mathrm{d} & 0\end{bmatrix}\begin{bmatrix}-1\\\delta_q\mathcal{C}\\\delta_\phi\mathcal{C}\end{bmatrix}$$
$$\delta_q\mathcal{C}\mid_0 = 1.$$

This yields a functional $\mathcal{C}$of the form $\int \tilde{q}(z,t)dz - \xi$. Hence, we have $\xi = \int \tilde{q}(z,t)dz + c$, with c some constant. We then have the candidate Lyapunov function of the form

$$\mathcal{V}(\chi) = \frac{1}{2}\int\left(\frac{\tilde{q}^2}{C} + \frac{\tilde{\phi}^2}{L}\right)dz + \int -\frac{\bar{q}}{C}.\tilde{q}dz + c,$$

and by setting $c = \frac{1}{2}\frac{\bar{q}}{C}^2$, we have the incremental Lyapunov function

$$\mathcal{V}(\chi) = \frac{1}{2}\int_Z\left(\frac{(q-\bar{q})^2}{C} + \frac{\phi^2}{L}\right)dz.$$

It follows that

$$\dot{\mathcal{V}}(\chi) = f_l e_l,$$

and in addition if we add the following boundary control law (damping injection through the boundary),

$$f_l = -Re_l, \quad R > 0,$$

we would then have

$$\dot{\mathcal{V}}(\chi) \leq 0,$$

and can use Proposition 5.20 to prove asymptotic stability.

The shallow water equations

In this section we consider the case of a system where we have a Hamiltonian which is *non-quadratic*. To this end we use the example of the nonlinear shallow water equations in the $1-$dimensional spatial domain case. The dynamics of the nonlinear shallow water equations are given by

$$\begin{bmatrix}\partial_t h\\\partial_t u\end{bmatrix} = \begin{bmatrix}0 & \mathrm{d}\\\mathrm{d} & 0\end{bmatrix}\begin{bmatrix}\delta_h H = *h*u\\\delta_u H = \frac{1}{2}*u*u + gh\end{bmatrix},$$

together with the boundary variables $(hu)\mid_b$and $(\frac{1}{2}u^2 + gh)\mid_b$. The total energy of the system is given by

$$\mathcal{H}(h,u) = \frac{1}{2}\int(\tilde{h}\tilde{u}^2 + g\tilde{h}^2)dz,$$

and it satisfies the energy balance

$$\frac{dH}{dt} = (\tilde{h}\tilde{u})(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \mid_0^l .$$

We now study the stability of the system at a desired (or forced) height and a zero velocity, i.e. at a point $(\bar{h}, 0)$, in the same way as stabilizing a transmission line at a *forced* voltage and zero current. We again consider a constant input at one of its boundary. The objective again is to look for candidate Lyapunov functions of the form

$$\mathcal{V}(\chi) = H(h,u) + H_s(\xi), \quad H_s(\xi) = -\bar{u}\xi,$$

with $\bar{u}$ the constant input (the desired or the forced height $\bar{h}$. where we wish to stabilize our system). The resulting dynamics of the interconnected system would be

$$\begin{aligned}
\begin{bmatrix} \dot{\xi} \\ \partial_t h \\ \partial_t u \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \cdot \mid_0 \\ 0 & 0 & \mathrm{d} \\ 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_s}{\partial \xi} \\ \delta_h H \\ \delta_u H \end{bmatrix} \\
\delta_h H \mid_0 &= -\frac{\partial H_s}{\partial \xi} \\
f_l &= \delta_h H \mid_l ; e_l = \delta_u H \mid_l .
\end{aligned}$$

The total energy is given by

$$H_{cl}(q,p,\xi) = \frac{1}{2}\int \tilde{h}\tilde{u}^2 + g\tilde{h}^2 - \bar{u}\xi,$$

with $H_c(\xi) = -\bar{u}\xi$ being the energy (unbounded) of the source system. Note that $\bar{u}$ corresponds to the constant input and does not in any way relate to the velocity u. The composed system then admits Casimirs of the form

$$F(h,u,\xi) = \int \tilde{h}(x,t)dz - \xi,$$

which yields the Lyapunov function

$$\mathcal{V}(\chi) = \frac{1}{2}\int (\tilde{h}\tilde{u}^2 + g\tilde{h}^2)dz - \bar{u}(\int \tilde{h}(x,t)dz + c),$$

by choosing the constant input $\bar{u} = g\bar{h}$ and $c = -\frac{1}{2}\bar{h}$ we have the following incremental Lyapunov function

$$\mathcal{V}(\chi) = \frac{1}{2}\int \tilde{h}\tilde{u}^2 + g(\tilde{h} - \bar{h})^2.$$

The time derivative of this Lyapunov function would then satisfy

$$\dot{\mathcal{V}}(\chi) = f_l e_l,$$

we can then add the boundary control law

$$f_l = -\alpha e_l, \qquad \alpha > 0.$$

So that we have $\dot{\mathcal{V}}(\chi) \leq 0$ and hence can use Proposition 5.20 to prove asymptotic stability.

Remark 5.23. The constant source (the controller) in this case could be seen as a water reservoir with a given height and this is precisely the height at which we stabilize the ″plant″system. In this way we could also stabilize a series of interconnected canals all at the same height and the zero velocity.

6

Spatial discretization of the shallow water equations

> *"A love affair with a pet hypothesis can waste years of precious time."*
> - Peter Medawar.

In the previous chapters, we have seen how the framework of port-Hamiltonian systems can be used for modeling and analysis of infinite-dimensional systems, such as the $n-$dimensional wave equation, fluid dynamical systems, as well as the ideal transmission line. Hereto a special type of infinite-dimensional Dirac structure has been introduced, based on Stokes' theorem. Physically, this Stokes-Dirac structure captures the basic balance laws of the system, like charge and the flux conservation or mass balance. We have seen in Chapter 2 how the port-Hamiltonian formulation is a non-trivial extension of the Hamiltonian formulation of partial differential equations (PDEs) by means of Poisson structures (see e.g. [34]), as in the latter case it is crucially assumed that the boundary conditions are such that the energy-flow through the boundary of the spatial domain is zero. We have also seen how the formulation in terms of Dirac structures, instead of a Poisson structure, allows a non-zero boundary energy-flow.

As stated in the introduction, one of the motivations to consider spatial discretization of an infinite-dimensional system is the interconnection of mixed finite and infinite-dimensional port-Hamiltonian systems. In Section 5.2.5 we considered the problem of stabilizing flow of water though a canal at a constant height and the zero velocity. The control action was a *constant* input at one of the boundary. Damping was then added through the boundary of the system to ensure asymptotic stability. Both these elements, the constant input and the dissipator at the boundary, are modeled as finite-dimensional systems. From the control point of view of such (mixed finite and infinite-dimensional) systems, it may be crucial to approximate the infinite-dimensional subsystem with a finite-dimensional one. The finite-dimensional approximation should be such that it is again a port-Hamiltonian systems which retains all the properties of the infinite-dimensional model, like energy balance and other conserved quantities. Furthermore the approximation should also take into account the constraints arising due to the interconnection of the infinite-dimensional system with finite-dimensional systems though the boundary, such that interconnected system (with the finite-dimensional approximation of the infinite-dimensional system) again has the port-Hamiltonian structure. It has been shown in [18] how the intrinsic Hamiltonian formulation suggests finite element methods which result in finite-dimensional approximations which are again port-Hamiltonian systems. Given the port-Hamiltonian formulation of distributed parameter systems it is natural to use *different* finite-elements for the approximation of functions and forms.

In this chapter, we consider spatial discretization of the shallow water equations, which are modeled as port-Hamiltonian systems in Chapter 2. We consider spatial discretization of the system defined by (2.63) (a *constant* Stokes-Dirac structure) and (2.69) (a *non-constant* Stokes-Dirac structure). We also present some preliminary numerical results for the constant Dirac structure case.

6.1 Spatial discretization of a Stokes-Dirac structure with 1-D spatial domain.

In this section we consider spatial discretization of infinite-dimensional port-Hamiltonian systems with a 1-D spatial domain, which is defined with respect to the following *constant* Stokes-Dirac structure. In particular we consider the case of spatial discretization of the shallow water equations, which were modeled as infinite-dimensional port-Hamiltonian systems in Chapter 2

$$
\begin{aligned}
\begin{bmatrix} f_h \\ f_u \end{bmatrix} &= \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_u \end{bmatrix} \\
\begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_h\ |\partial Z \\ e_u\ |\partial Z \end{bmatrix}, \\
f_h &= -\frac{\partial h}{\partial t},\ \ f_u = -\frac{\partial u}{\partial t} \\
e_h &= \delta_h \mathcal{H},\ e_u = \delta_u \mathcal{H},
\end{aligned}
$$

with h being the height of the water level, u the velocity and $\mathcal{H}$ the Hamiltonian (the total energy of the system).

Our interest is in the discretization of the shallow water equations and we focus on two different cases, namely 1) The linear shallow water equations and 2) The nonlinear shallow water equations. In both the cases the interconnection structure remains the same and what changes is the expression for the total energy. First we discuss the discretization of interconnection structure and then their respective Hamiltonians

The spatial discretization proposed here consists of two steps. First, the interconnection structure of the distributed parameter model is spatially discretized Next, the constitutive relations of the energy storage part of the system to be discretized are approximated.

6.1.1 Tessellation

At first, we have to tessellate the spatial domain with cells or elements denoted as Z_{ab} with spatial manifold $[S_{i-1}, S_i]$ such that

$$Z = \{\bigcup_{1}^{N} Z_{ab}, 1 \leq i \leq N+1, 0 < S_{i-1} < S_i < l\}.$$

It is convenient to introduce a reference element $\hat{Z}$ with the spatial manifold $[-1, 1]$ such that each element Z_{ab} is mapped to the reference element $\hat{Z}$ using the mapping

$$F_Z : \hat{Z} \to Z_{ab} : z = \frac{1}{2}\big(S_{i-1}(1-\zeta) + S_i(1+\zeta)\big),$$

where ζ is the coordinate of the reference element $\hat{Z}$.

6.1.2 Spatial discretization of the interconnection structure

Consider a part of the canal between two points a and b $(0 \leq a \leq b \leq L)$. The spatial manifold corresponding to this part of the canal is $Z_{ab} = [a, b]$. The fluid flow at the point a is defined f_a^B and the Bernoulli function is denoted by e_a^B. Similarly with the fluid flow and the Bernoulli function at point b.The relations between the boundary variables $f_a^B, e_a^B, f_b^B, e_b^B$ and the efforts e_h, e_u are

$$e_a^B(t) = e_h(t, a), \quad e_b^B(t) = e_h(t, b),$$

$$f_a^B(t) = e_u(t, a), \quad f_b^B(t) = e_u(t, b).$$

The discretization method follows similar procedure as that of an ideal transmission line as in [18].

Approximation of f_h and f_u: The infinitesimal height f_h and f_u are approximated on Z_{ab} as

$$\begin{aligned} f_h(t, z) &= f_{ab}^h(t)\omega_{ab}^h(z) \\ f_u(t, z) &= f_{ab}^u(t)\omega_{ab}^u(z), \end{aligned} \tag{6.1}$$

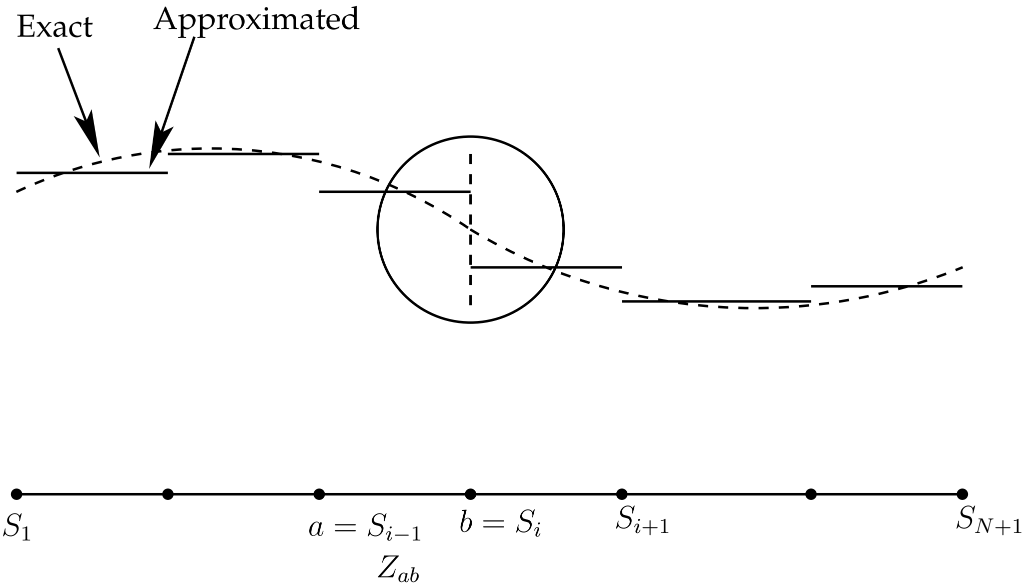


Figure 6.1: An illustration of the approximation of a flow variable.

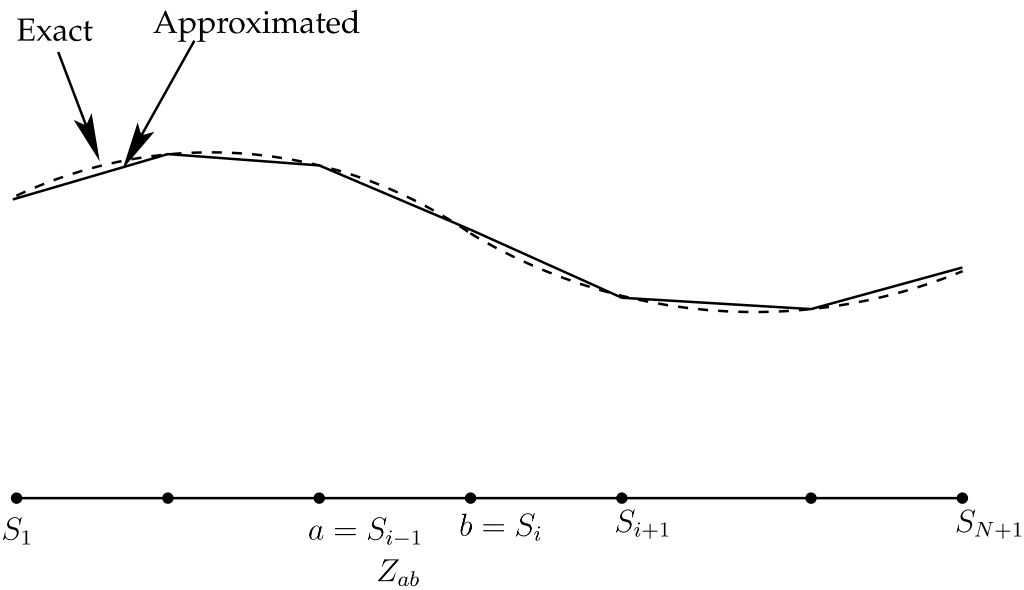


Figure 6.2: An illustration of the approximation of an effort variable.

where the one-forms ω^h_{ab} and ω^u_{ab} satisfy

$$\int_{Z_{ab}} \omega^h_{ab} = 1, \quad \int_{Z_{ab}} \omega^u_{ab}(z) = 1. \tag{6.2}$$

Since the flow variables are approximated per element they can be discontinuous across the nodes of the elements and hence they are multivalued at the nodes as illustrated in Figure 6.1.
Approximation of e_h and e_u: The co-energy variables $e_h(z,t)$ and $e_u(z,t)$ are approximated as

$$e_h(t,z) = e^h_a(t)\omega^h_a(z) + e^h_b(t)\omega^h_b(z) \tag{6.3}$$
$$e_u(t,z) = e^u_a(t)\omega^u_a(z) + e^u_b(t)\omega^u_b(z), \tag{6.4}$$

where the zero-forms $\omega^h_a, \omega^h_b, \omega^u_a\omega^b_u \in \Omega^0(Z_{ab})$ satisfy

$$\begin{aligned} &\omega^h_a(a) = 1, \quad \omega^h_a(b) = 0, \quad \omega^h_a(a) = 0, \quad \omega^h_b(b) = 1, \\ &\omega^u_a(a) = 1, \quad \omega^u_a(b) = 0, \quad \omega^u_b(a) = 0, \quad \omega^u_b(b) = 1. \end{aligned} \tag{6.5}$$

such that the efforts are always continuous across the edges of the elements as illustrated in Figure 6.2.

This gives

$$f^h_{ab}(t)\omega^h_{ab}(z) = e^u_a(t)\mathrm{d}\omega^u_a + e^u_b(t)\mathrm{d}\omega^u_b(z) \tag{6.6}$$
$$f^u_{ab}(t)\omega^u_{ab}(z) = e^h_a(t)\mathrm{d}\omega^h_a + e^h_b(t)\mathrm{d}\omega^h_b(z). \tag{6.7}$$

Compatibility of forms: (1) The one form $\omega^h_{ab}(z)$ and functions $\omega^u_a(z)$ and $\omega^u_b(z)$ should be chosen in such a way that for every e^u_a, e^u_b we can find f^h_{ab} such that (6.6) is satisfied.
(2) The one form $\omega^u_{ab}(z)$ and functions $\omega^h_a(z)$ and $\omega^h_b(z)$ should be chosen in such a way that for every e^h_a, e^h_b we can find f^{ab}_u such that (6.7) is satisfied. The above compatibility conditions imply the following relations between the one form $\omega^h_{ab}(z)$ and the functions $\omega^u_a(z)$ and $\omega^u_b(z)$ chosen in the approximation of e_h and e_u. Take $e^u_b = 0$. Then (6.6) is true if and only if $\mathrm{d}\omega^u_a = c\omega^h_{ab}$ for a constant c. Integrating this over Z_{ab} yields $\omega^u_a(b) - \omega^u_a(a) = c\int_{Z_{ab}} \omega^h_{ab}(z)$. From (6.2) and (6.6) we have $c = -1$. Therefore

$$\mathrm{d}\omega^u_a = -\omega^h_{ab}. \tag{6.8}$$

By choosing $e^u_a = 0$, we can prove that

$$\mathrm{d}\omega^b_u = \omega^h_{ab}. \tag{6.9}$$

Using similar arguments it can also be shown that

$$\mathrm{d}\omega^h_a = -\omega^u_{ab}, \mathrm{d}\omega^h_b = \omega^{ab}_u. \tag{6.10}$$

As a consequence of the compatibility conditions the functions $\omega_a^h, \omega_b^h, \omega_a^u, \omega_b^u$ are completely determined by the one-forms ω_{ab}^h and ω_{ab}^u. We also have the following properties of the corresponding zero and one forms:

Proposition 6.1. *[18]$\omega_a^h, \omega_b^h, \omega_a^u, \omega_u^b, \omega_{ab}^h$ and ω_{ab}^u satisfy*

1) $\omega_a^h(z) + \omega_b^h(z) = 1.$

2) $\omega_a^u(z) + \omega_u^b(z) = 1.$

3) $\int_{Z_{ab}} \omega_a^h(z)\omega_{ab}^h(z) + \int_{Z_{ab}} \omega_b^h(z)\omega_{ab}^h(z) = 1.$

4) $\int_{Z_{ab}} \omega_a^u(z)\omega_{ab}^u(z) + \int_{Z_{ab}} \omega_b^u(z)\omega_{ab}^u(z) = 1.$

5) $\int_{Z_{ab}} \omega_a^h(z)\omega_{ab}^h(z) + \int_{Z_{ab}} \omega_b^u(z)\omega_{ab}^u(z) = 1.$

Proof. From, $\mathrm{d}(\omega_a^h(z) + \omega_b^h(z)) = 0 \Rightarrow \omega_a^h(z) + \omega_b^h(z) = \omega_a^h(0) + \omega_b^h(0) = 1,$ from (6.5).
2) From (6.8,6.9) $\mathrm{d}(\omega_a^u(z) + \omega_b^u(z)) = 0 \Rightarrow \omega_a^u(z) + \omega_b^u(z) = \omega_a^u(0) + \omega_b^u(0) = 1,$ from(6.5).
3) $\int_{Z_{ab}} \omega_a^h(z)\omega_{ab}^h(z) + \int_{Z_{ab}} \omega_b^h(z)\omega_{ab}^h(z) = 1 = \int_{Z_{ab}} (\omega_a^h(z) + \omega_b^h(z))\omega_a^h(z) = \int \omega_a^h(z) = 1$, from (6.2).
4) $\int_{Z_{ab}} \omega_a^u(z)\omega_{ab}^u(z) + \int_{Z_{ab}} \omega_b^u(z)\omega_{ab}^u(z) = \int_{Z_{ab}} (\omega_a^u(z) + \omega_b^u(z))\omega_{ab}^u(z) = \int_{Z_{ab}} \omega_{ab}^u(z) = 1.$
5) $\int_{Z_{ab}} \omega_a^h(z)\omega_{ab}^h(z) + \int_{Z_{ab}} \omega_b^u(z)\omega_{ab}^u(z) = -\int d(\omega_a^h(z)\omega_a^h(z)) = \omega_a^h(a)\omega_a^u(a) - \omega_a^h(b)\omega_a^u(b)$, from (6.5). □

The discretized interconnection structure is then obtained as follows: By substituting (6.8) and (6.9) into (6.6), we get

$$f_{ab}^h(t) = e_a^u(t) - e_b^u(t), \tag{6.11}$$

similarly

$$f_{ab}^u(t) = e_a^h(t) - e_b^h(t). \tag{6.12}$$

The following expression describes the spatially discretized interconnection structure of the considered part of the canal

$$\begin{bmatrix} e_a^B \\ e_b^B \\ f_a^B \\ f_b^B \\ f_{ab}^h \\ f_{ab}^u \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_a^h \\ e_b^h \\ e_a^u \\ e_b^h \end{bmatrix}. \tag{6.13}$$

The net power of the considered part of the canal is

$$\int_{Z_{ab}} e_h(t) f_h(t) + \int_{Z_{ab}} e_u(t) f_u(t) - e_a^B f_a^B + e_b^B f_b^B. \tag{6.14}$$

We then have

$$P_{ab}^{net} = [\alpha_{ab} e_a^h + (1-\alpha_{ab}) e_b^h] f_{ab}^h + [(1-\alpha_{ab}) e_a^u + \alpha_{ab} e_b^u] f_{ab}^u - e_a^B f_a^B + e_b^B f_b^B, \tag{6.15}$$

where $\alpha_{ab} := \int_{Z_{ab}} \omega_a^h(z) \omega_{ab}^h(z)$. The above expression is used for identifying the port variables in the discretized interconnection structure. The flow variable corresponding to the mass density is f_{ab}^h and the effort variable is $[\alpha_{ab} e_a^h + (1-\alpha_{ab}) e_b^h]$ and similarly the flow variable corresponding to the velocity is f_{ab}^u and the corresponding effort variable is $[(1-\alpha_{ab}) e_a^u + \alpha_{ab} e_b^u]$. Thus by defining

$$\begin{aligned} e_{ab}^h &:= [\alpha_{ab} e_a^h + (1-\alpha_{ab}) e_b^h] \\ e_{ab}^u &:= [(1-\alpha_{ab}) e_a^u + \alpha_{ab} e_b^u], \end{aligned} \tag{6.16}$$

and the expression for P_{ab}^{net} becomes

$$P_{ab}^{net} :=< e_{ab} \mid f_{ab} >= f_{ab}^h e_{ab}^h + f_u^{ab} e_{ab}^u - e_a^B f_a^B + e_b^B f_b^B. \tag{6.17}$$

This gives

$$\begin{bmatrix} -1 & 0 & \alpha_{ab} & \alpha_{ba} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} e_{ab}^h \\ e_{ab}^u \\ e_a^B \\ e_b^B \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{ba} & \alpha_{ab} \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{ab}^h \\ f_{ab}^u \\ f_a^B \\ f_b^B \end{bmatrix} = 0, \tag{6.18}$$

where $\alpha_{ba} = 1 - \alpha_{ab}$. The above equation represents the spatially discretized interconnection structure, abbreviated as

$$D_{ab} = \{(f_{ab}, e_{ab}) \in \mathbb{R}^8 : E_{ab} e_{ab} + F_{ab} f_{ab} = 0\}. \tag{6.19}$$

It can easily be shown that the above subspace D_{ab} is a Dirac structure with respect to the bilinear form

$$<< (f_{ab}^1, e_{ab}^1), (f_{ab}^2, e_{ab}^2) >>:=< e_{ab}^1, f_{ab}^2 > + < e_{ab}^2, f_{ab}^1 > . \tag{6.20}$$

6.1.3 Approximation of the energy part

After the spatial discretization of the interconnection structure, the next step is to discretize the constitutive relations of the energy storage. Recall that in the port-Hamiltonian representation the system is specified by its Dirac structure, the interconnection structure, together with its Hamiltonian, the constitutive relations of the energy storage. We now discuss both of the above mentioned cases individually.

The linear shallow water equations

We consider the dynamics of the shallow water equations, linearized around a height h_l and zero velocity, i.e. around $(h_l, 0)$. These are derived as follows: Consider small variations of the system around $(h_l, 0)$ and denote

$$\begin{aligned} h(t,z) &= h_l + \epsilon\eta(t,z) \\ u(t,z) &= \epsilon u'(t,z). \end{aligned} \tag{6.21}$$

Substituting (6.21) in the shallow water equations given by Equation (2.64) and taking the limit as $\epsilon \to 0$, we get the following

$$\begin{aligned} \partial_t \eta + \mathrm{d}(h_l * u') &= 0 \\ \partial_t u' + \mathrm{d}(g * \eta) &= 0. \end{aligned} \tag{6.22}$$

For sake of consistent notation throughout, we replace η in (6.22) by h and u' by u. This gives us the following linear shallow water equations

$$\begin{aligned} \partial_t h + \mathrm{d}(h_l * u) &= 0 \\ \partial_t u + \mathrm{d}(g * h) &= 0. \end{aligned}$$

The total energy in this case is given by

$$\mathcal{H} = \frac{1}{2}\int_Z [h_l(*u)u + g(*h)h].$$

Both the flow variables f^h and f^u and the energy variables h and u are one-forms. Since f^h and f^u are approximated by (6.1), and they are related to h and u by (6.11) and (6.12), it is consistent to approximate h and u on Z_{ab} in the same way by

$$\begin{aligned} h(t,z) &= h_{ab}(t)\omega^h_{ab}(z) \\ u(t,z) &= u_{ab}(t)\omega^u_{ab}(z), \end{aligned} \tag{6.23}$$

where

$$-\tfrac{dh_{ab}(t)}{dt} = f^h_{ab}(t), \quad -\tfrac{du_{ab}(t)}{dt} = f^u_{ab}(t) \ . \tag{6.24}$$

Observe that h_{ab} represents the total amount of water in the considered part of the canal and u_{ab} represents the velocity in the same part. The kinetic energy as a function of the energy variable u is given by $\int_{Z_{ab}} \frac{1}{2} h_l(*u(t,z)u(t,z)$. Approximation of the infinite-dimensional energy variable u by (6.23) means that we restrict the infinite-dimensional space of one-forms $\Omega^1(Z_{ab})$ to its one-dimensional subspace spanned by ω^u_{ab}. This leads to the approximation of the kinetic energy of the considered part of the canal by

$$H^u_{ab}(u_{ab}(t)) = \frac{C_1}{2} h_l u_{ab}^2,$$

where

$$C_1 = \int_{Z_{ab}} (*\omega^u_{ab}(z))\omega^u_{ab}(z).$$

Note that this is nothing else than the restriction of the kinetic energy function to the one-dimensional subspace of $\Omega^1(Z_{ab})$ spanned by $\omega^u_{ab}(z)$. Similarly the potential energy is approximated by

$$H^h_{ab}(h_{ab}(t)) = \frac{C_2}{2} g h^2_{ab},$$

where

$$C_2 = \int_{Z_{ab}} (*\omega^h_{ab}(z))\omega^h_{ab}(z).$$

Therefore the total energy of the considered part of the canal is approximated by

$$\begin{aligned} H_{ab}(h_{ab}, u_{ab}) &= H^u_{ab}(u_{ab}) + H^h_{ab}(h_{ab}) \\ &= \frac{1}{2}(C_1 h_l u^2_{ab} + C_2 g h^2_{ab}). \end{aligned}$$

In order to describe the discretized dynamics, we equate the discretized effort variables e^h_{ab}, e^u_{ab} of the discretized interconnection structure defined in (6.16) with co-energy variables corresponding to the total approximated energy H_{ab} of the considered part of the canal

$$\begin{aligned} e^h_{ab} &= \tfrac{\partial H(h_{ab},u_{ab})}{\partial h_{ab}}(t) = C_2 g h_{ab} \\ e^u_{ab} &= \tfrac{\partial H(h_{ab},u_{ab})}{\partial u_{ab}}(t) = C_1 h_l u_{ab}. \end{aligned} \tag{6.25}$$

The equations (6.18) (the interconnection structure) together with (6.24),(6.25) represent a finite dimensional model of the shallow water equations. To sum-up we obtain the following set of DAEs for a single lump of the finite-dimensional model

$$\begin{aligned} &-\tfrac{dh_{ab}}{dt} = h_l u \mid_a - h_l u \mid_b \\ &-\tfrac{du_{ab}}{dt} = gh \mid_a - gh \mid_b \\ &C_2 g h_{ab} = \alpha_{ab}(gh \mid_a) + \alpha_{ba}(gh \mid_b) \\ &C_1 h_l u_{ab} = \alpha_{ba}(h_l u \mid_a) + \alpha_{ab}(h_l u \mid_b). \end{aligned} \tag{6.26}$$

Spatial discretization of the linear shallow water equations

The canal is split into n parts. The ith part (S_{i-1}, S_i) is discretized as explained in the previous subsections, where $a = S_{i-1}$ and $b = S_i$. The resulting model consists of n sub models each of them representing a port-

Hamiltonian system. Since a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system, the total discretized system is also a port-Hamiltonian system, whose interconnection structure is given by the composition of the n Dirac structures on (S_{i-1}, S_i), while the total Hamiltonian is the sum of individual Hamiltonians.

$$H(h,u) = \sum_{i=1}^{n} [C_{1i} h_l u_{S_{i-1},S_i} + C_{2i} g h^2_{S_{i-1},S_i}].$$

Here $h = (h_{S_0,S_1}, h_{S_1,S_2}, ..., h_{S_{n-1},S_n})^T$ are the discretized heights and $u = (u_{S_0,S_1}, u_{S_1,S_2}, ..., u_{S_{n-1}}, u_{s_n})$ are the discretized velocities. The total discretized model still has two ports. The port $(f^B_{S_0}, e^B_{S_0}) = (f^B_0, e^B_0)$ is the incoming port and the port $(f^B_{S_n}, e^B_{Sn}) = (f^B_S, e^B_S)$ is the outgoing port, resulting in the energy balance of the discretized model

$$\frac{dH(h(t), u(t))}{dt} - e^B_0 f^B_0 + e^B_S f^B_S = 0.$$

Equation (6.11) for the ith part becomes $f^h_{S_{i-1},S_i}(t) = e^u_{S_{i-1}}(t) - e^u_{S_i}(t)$. Taking into account (6.24) and $e^u_{S_0} = f^B_0$, $e_{S_n} = f^B_S$, we have $\frac{dh(t)}{dt} = f^B_0 - f^B_S$, where $h := \sum_{i=1}^{n} h_{S_{i-1},S_i}$ is the total mass (amount of water) in the canal, this represents mass conservation. Another conserved quantity is the velocity, which is obtained from (6.12), (6.24), i.e. $\frac{du(t)}{dt} = e^B_0 - e^B_S$, where $u = \sum_{i=1}^{n} u_{S_{i-1},S_i}$, represents the average velocity in the canal.

The nonlinear shallow water equations

The dynamics of the nonlinear shallow water equations are given by (2.64)

$$\begin{aligned} \partial_t h + \mathrm{d}(hu) &= 0 \\ \partial_t u + \mathrm{d}(gh) &= 0. \end{aligned}$$

The total energy in this case is given by

$$H = \frac{1}{2} \int_Z [(*u)h(*u) + g(*h)h].$$

As before, the flow variables f_h and f_u and the energy variables h and u are one-forms and they are approximated in the same way as in (6.24). In the nonlinear case, the kinetic energy as a function of the energy variable u is given by $\int_{Z_{ab}} \frac{1}{2}(*u(t,z)h(*u(t,z))$. Approximation of the infinite-dimensional energy variable u by (6.23) means that we restrict the infinite-dimensional space of one-forms $\Omega^1(Z_{ab})$ to its one-dimensional subspace spanned by ω^u_{ab}.

This leads to the approximation of the kinetic energy of the considered part of the canal by

$$H^u_{ab}(u_{ab}(t)) = \frac{1}{2}C_1 h_{ab} u^2_{ab},$$

where

$$C_1 = \int_{Z_{ab}} (*\omega^u_{ab}(z))\omega^u_{ab}(z)(*\omega^u_{ab}(z)).$$

Note that this is nothing else than the restriction of the kinetic energy function to the one-dimensional subspace of $\Omega^1(Z_{ab})$ spanned by $\omega^h_{ab}(z)$. Similarly the potential energy is approximated by

$$H^h_{ab}(h_{ab}(t)) = \frac{1}{2}C_2 g h^2_{ab},$$

where

$$C_2 = \int_{Z_{ab}} (*\omega^h_{ab}(z))\omega^h_{ab}(z).$$

Therefore the total energy of the considered part of the canal is approximated by

$$\begin{aligned} H_{ab}(h_{ab}, u_{ab}) &= H^u_{ab}(u_{ab}) + H^h_{ab}(h_{ab}) \\ &= \frac{1}{2}[C_1 h_{ab} u^2_{ab} + C_2 g h^2_{ab}]. \end{aligned}$$

In order to describe the discretized dynamics, we equate the discretized effort variables e^h_{ab}, e^u_{ab} of the discretized interconnection structure defined in (6.16) with co-energy variables corresponding to the total approximated energy H_{ab} of the considered part of the canal

$$\begin{aligned} e^h_{ab} &= \tfrac{\partial H(h_{ab},u_{ab})}{\partial h_{ab}}(t) = \tfrac{C_1}{2}u^2_{ab} + C_2 g h_{ab} \\ e^u_{ab} &= \tfrac{\partial H(h_{ab},u_{ab})}{\partial u_{ab}}(t) = C_1 h_{ab} u_{ab}. \end{aligned} \tag{6.27}$$

The equations (6.18) (the interconnection structure) together with (6.24),(6.27) represent a finite dimensional model of the non-linear shallow water equations. To sum-up we obtain the following set of DAEs for a single lump of the finite-dimensional model

$$\begin{aligned} &-\tfrac{dh_{ab}}{dt} = hu\mid_a - hu\mid_b \\ &-\tfrac{du_{ab}}{dt} = (\tfrac{1}{2}u^2 + gh)\mid_a - (\tfrac{1}{2}u^2 + gh)\mid_b \\ &\tfrac{C_1}{2}u^2_{ab} + C_2 g h_{ab} = \alpha_{ab}(\tfrac{1}{2}u^2 + gh\mid_a) + \alpha_{ba}(\tfrac{1}{2}u^2 + gh\mid_b) \\ &C_1 h_{ab} u_{ab} = \alpha_{ba}(hu\mid_a) + \alpha_{ab}(hu\mid_b). \end{aligned} \tag{6.28}$$

Spatial discretization of the nonlinear shallow water equations

The spatial discretization follows the same procedure as in the case of linear shallow water equations. The total Hamiltonian is now given by

$$H(h,u)=\sum_{i=1}^{n}[C_{1i}h_{S_{i-1},S_i}u_{S_{i-1},S_i}+C_{2i}gh^2_{S_{i-1},S_i}],$$

resulting in the energy balance of the discretized model as

$$\frac{dH(h(t),u(t))}{dt}-e_0^Bf_0^B+e_S^Bf_S^B=0,$$

with (f_0^B,e_0^B) the incoming port and (f_S^B,e_S^B) the outgoing port. It can also be easily verified that the mass and the velocity conservation laws also hold in the case of the nonlinear shallow water equations.

The input state output model

In this section we write the discretized system in the input state output port-Hamiltonian model for a single spatial element. For our analysis we use the following choices for the approximating zero and one forms. The zero-forms are approximated as constant density functions, i.e.

$$\omega_{ab}^{h,u}=\frac{1}{b-a},$$

and the zero-forms as linear splines, i.e.

$$\omega_a^{h,u}=\frac{b-z}{b-a},\ \ \omega_b^{h,u}=\frac{z-a}{b-a}.$$

This would result in the following values for the constants in (6.44).

$$\alpha_{ab}=\alpha_{ba}=\frac{1}{2}.$$

We then have the following

$$\begin{aligned}\begin{bmatrix}f_{ab}^h\\ f_{ab}^u\end{bmatrix}&=\begin{bmatrix}0&-2\\2&0\end{bmatrix}\begin{bmatrix}e_{ab}^h\\ e_{ab}^u\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}f_a^B\\-e_b^B\end{bmatrix}\\ \begin{bmatrix}e_a^B\\ f_b^B\end{bmatrix}&=\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}e_{ab}^h\\ e_{ab}^u\end{bmatrix}+\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}f_a^B\\-e_b^B\end{bmatrix}.\end{aligned}\tag{6.29}$$

The above equation is an input output representation of (6.18), with inputs $(f_a^B,-e_b^B)$ and outputs (e_a^B,f_b^B).

Discussion

It is interesting to note here that if we consider a periodic domain then we should only consider odd number of spatial lumps. The reason for this can be explained as follows: Consider a periodic domain with only two lumps, with the input state output model (6.29) of each of the lump in the following form

$$\begin{aligned}\dot{x}_i &= A_i x + B_i u_i \\ y_i &= C_i x + D_i u_i,\ i = 1, 2.\end{aligned}$$

In terms of the discretized shallow water equations

$$A_i = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$C_i = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, D_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We know from system theory that an interconnection of the two such lumps, in the standard plant controller interconnection constraints

$$\begin{aligned}u_1 &= -y_2 \\ u_2 &= y_1,\end{aligned}$$

is well-posed if and only if the following two conditions are satisfied

$$\begin{aligned}&\det[I + D_1 D_2] \neq 0 \\ &\det[I + D_2 D_1] \neq 0.\end{aligned} \tag{6.30}$$

It can easily be seen that for the input state output model of the shallow water equations, the above conditions are not satisfied and hence we cannot interconnect the two spatially distributed lumps. Same is the case if we consider any even number of lumps. This is not the case if we consider odd number of lumps. Hence its crucial that in our analysis we consider only an odd number of lumps for a periodic domain.

For the sake of illustration, of an input output model, we present a case where we have a periodic domain and we take three spatial lumps, i.e. $n = 3$. In this case we would have four nodes and since the domain is periodic the fourth and the first nodes are the same and hence the port-variables at the 1st and the 4th node would hold same values. The dynamics of the discretized system would then be given by the following

$$\begin{bmatrix} f^h_{S_1,S_2} \\ f^h_{S_2,S_3} \\ f^h_{S_3,S_1} \\ f^u_{S_1,S_2} \\ f^u_{S_2,S_3} \\ f^u_{S_3,S_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^h_{S_1,S_2} \\ e^h_{S_2,S_3} \\ e^h_{S_3,S_1} \\ e^u_{S_1,S_2} \\ e^u_{S_2,S_3} \\ e^u_{S_3,S_1} \end{bmatrix}. \tag{6.31}$$

where

$$f^h_{S_iS_j} = -\frac{dh_{S_i,S_j}}{dt}, \qquad f^h_{S_iS_j} = -\frac{dh_{S_i,S_j}}{dt},$$

$$e^h_{S_iS_j} = \frac{\partial H(h_{S_i,S_j}, u_{S_i,S_j})}{\partial h_{S_i,S_j}}, \quad e^u_{S_iS_j} = \frac{\partial H(h_{S_i,S_j}, u_{S_i,S_j})}{\partial u_{S_i,S_j}}, \quad i,j = 1,2,3;\ i \neq j.$$

The conservation of energy follows from the skew-symmetry of the 6×6 interconnection matrix in (6.31), in other words we have

$$\frac{dH}{dt}(h,u) = 0.$$

since the domain is periodic (or *closed*).

6.2 Spatial discretization of a *non-constant* Dirac structure

So far we have discussed spatial discretization of a constant Stokes-Dirac structure, where the Dirac structure does not depend on the energy variables. In this section we discuss spatial discretization of a non-constant Stokes-Dirac structure, where the Dirac structure now depends on the energy variables. To this end we use the example of the shallow water equations with a additional velocity component as discussed in Chapter 2, Equation (2.69). The Dirac structure is given by the following

$$\begin{aligned}\begin{bmatrix} f_h \\ f_u \\ f_v \end{bmatrix} &= \begin{bmatrix} 0 & \mathrm{d} & 0 \\ \mathrm{d} & 0 & -\frac{1}{*h}\mathrm{d}(*v) \\ 0 & \frac{1}{*h}\mathrm{d}(*v) & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_u \\ e_v \end{bmatrix} \\ \begin{bmatrix} f_b \\ e_b \\ e'_v \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{*h} \end{bmatrix} \begin{bmatrix} e_u\ |_{\partial W} \\ e_h\ |_{\partial W} \\ e_v\ |_{\partial W} \end{bmatrix},\end{aligned} \tag{6.32}$$

with

$$\begin{aligned} f_h &= -\tfrac{\partial h}{\partial t}, \quad f_u = -\tfrac{\partial u}{\partial t}, \quad f_v = -\tfrac{dv}{dt} \\ e_h &= \delta_h \mathcal{H}, \quad e_u = \delta_u \mathcal{H}, \quad e_v = \delta_v \mathcal{H} \end{aligned}.$$

6.2.1 Spatial discretization of the interconnection structure

Consider a part of the canal between two points a and b $(0 \leq a < b \leq L)$. The spatial manifold corresponding to this part of the canal is $Z_{ab} = [a,b]$. The mass flow through point a is denoted by e^B_a and the Bernoulli function by f^B_a, similarly for the point b with e^B_b and f^B_b respectively.

Approximation of f_h,f_u and f_v: As in the case of a *constant* Dirac structure, we approximate the infinitesimal height f_h, the velocities f_u, f_v on Z_{ab} as

$$\begin{aligned} f_h(t,z) &= f^h_{ab}(t)\omega^h_{ab}(z) \\ f_u(t,z) &= f^u_{ab}(t)\omega^u_{ab}(z) \\ f_v(t,z) &= f^v_{ab}(t)\omega^v_{ab}(z), \end{aligned} \tag{6.33}$$

where again the one-forms ω^h_{ab}, ω^u_{ab} and ω^v_{ab}satisfy

$$\int_{Z_{ab}} \omega^h_{ab} = 1, \quad \int_{Z_{ab}} \omega^u_{ab}(z) = 1, \quad \int_{Z_{ab}} \omega^v_{ab}(z) = 1. \tag{6.34}$$

Approximation of e_h and e_u: The co-energy variables $e_h(z,t)$ and $e_u(z,t)$ are approximated as

$$\begin{aligned} e_h(t,z) &= e^h_a(t)\omega^h_a(z) + e^h_b(t)\omega^b_h(z) \\ e_u(t,z) &= e^u_a(t)\omega^u_a(z) + e^u_b(t)\omega^b_u(z) \\ e_v(t,z) &= e^v_a(t)\omega^a_v(z) + e^v_b(t)\omega^v_b(z), \end{aligned} \tag{6.35}$$

where the zero-forms $\omega^h_a, \omega^h_b, \omega^u_a, \omega^u_b, \omega^v_a, \omega^v_b \in \Omega^0(Z_{ab})$ satisfy

$$\begin{array}{llll} \omega^h_a(a) = 1, & \omega^h_a(b) = 0, & \omega^h_b(a) = 0, & \omega^h_b(b) = 1, \\ \omega^u_a(a) = 1, & \omega^u_a(b) = 0, & \omega^u_b(a) = 0, & \omega^u_b(z) = 1, \\ \omega^v_a(a) = 1, & \omega^v_a(b) = 0, & \omega^v_b(a) = 0, & \omega^v_b(b) = 1. \end{array}$$

Furthermore we also approximate $*v(t,z)$ in (6.32) with a zero form as (instead of approximating $v(t,z)$ with a one form, see remarks towards the end of the section)

$$*v(t,z) = v_a(t)\omega_a(t) + v_b(t)\omega_b(t), \tag{6.36}$$

where the zero forms $\omega_a(z)$ and $\omega_a(z)$ satisfies

$$\begin{array}{ll} \omega_a(a) = 1, & \omega_a(b) = 0, \\ \omega_b(a) = 0, & \omega_b(b) = 1. \end{array}$$

this gives

$$f^h_{ab}(t)\omega^h_{ab}(z) = e^u_a(t)\mathrm{d}\omega^u_a(z) + e^u_b(t)\mathrm{d}\omega^u_b(z) \tag{6.37}$$

$$\begin{aligned} &f^u_{ab}(t)\omega^u_{ab}(z) = e^h_a(t)\mathrm{d}\omega^h_a(z) + e^h_b(t)\mathrm{d}\omega^h_b(z) - \\ &\frac{1}{h_{ab}(t) * \omega^h_{ab}(z)}(v_a(t)\mathrm{d}\omega_a(t) + v_b(t)\mathrm{d}\omega_b(t))(e^v_a(t)\omega^v_a(z) + e^v_b(t)\omega^v_b(z)) \end{aligned} \tag{6.38}$$

$$\begin{aligned} &f^v_{ab}(t)\omega^v_{ab}(z) = \\ &\frac{1}{h_{ab}(t) * \omega^h_{ab}(z)}(v_a(t)\mathrm{d}\omega_a(t) + v_b(t)\mathrm{d}\omega_b(t))(e^u_a(t)\omega^u_a(z) + e^u_b(t)\omega^u_b(z)), \end{aligned} \tag{6.39}$$

where the height $h(z,t)$ is approximated as

$$h(z,t) = h_{ab}(t)\omega_{ab}^h(z), \text{ where, } \int_{Z_{ab}} \omega_{ab}^h(z) = 1.$$

Compatibility of forms: In the first line of the above equation the one form ω_{ab}^h and the functions $\omega_a^u(z)$ and $\omega_u^b(z)$ should be chosen in such a way that for every e_a^u and e_b^u, we can find f_h^{ab} such that (6.37) is satisfied. The satisfaction of such conditions leads to the following equations

$$f_{ab}^h(t) = e_a^u(t) - e_b^u(t). \tag{6.40}$$

The above expression can also be obtained by integrating (6.37) over Z_{ab} and substituting the conditions on the zero and one forms (6.34,6.35). Similar satisfaction of compatibility conditions for (6.37) gives us the following equations

$$\begin{aligned} f_{ab}^u(t) &= e_a^h(t) - e_b^h(t) - \\ &\frac{1}{h_{ab}(t)}(c_1 v_a(t)e_a^v(t) + c_2 v_a(t)e_v^b(t) + c_3 v_b(t)e_a^v(t) + c_4 v_b(t)e_b^v(t)), \end{aligned} \tag{6.41}$$

where the constants are given by (again this is obtained by integrating over Z_{ab})

$$c_1 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega_{ab}^h}\omega_v^a, \qquad c_2 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega_{ab}^h}\omega_b^v,$$

$$c_3 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega_{ab}^h}\omega_v^a, \qquad c_4 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega_{ab}^h}\omega_b^v.$$

Similar satisfaction of compatibility conditions for (6.39) and integrating it over Z_{ab} yields

$$f_v^{ab}(t) = \frac{1}{h_{ab}(t)}(c_1' v_a(t)e_a^u(t) + c_2' v_a(t)e_b^u(t) + c_3' v_b(t)e_a^u(t) + c_4' v_b(t)e_b^u(t)),$$

where

$$c_1' = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega_{ab}^h}\omega_u^a, \qquad c_2' = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega_{ab}^h}\omega_b^u,$$

$$c_3' = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega_{ab}^h}\omega_u^a, \qquad c_4' = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega_{ab}^h}\omega_b^u.$$

For the sake of clarity the argument t is omitted in the rest of the section. The relations describing the spatially discretized interconnection structure of the part of the canal are given by

$$\begin{bmatrix} e_a^B \\ e_b^B \\ f_a^B \\ f_b^B \\ e_{va}^B \\ e_{vb}^B \\ f_{ab}^h \\ f_{ab}^u \\ f_{ab}^v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -k_1 & -k_2 \\ 0 & 0 & k_1 & k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_a^h \\ e_b^h \\ e_a^u \\ e_b^u \\ e_a^v \\ e_b^v \end{bmatrix}.$$

where $k_1 = \frac{1}{h_{ab}}(c_1 v_a + c_3 v_b)$ and $k_2 = \frac{1}{h_{ab}}(c_2 v_a + c_4 v_b)$.The net power in the considered part of the canal is

$$\int_{Z_{ab}} [e_h f_h + e_u f_u + e_v f_v] - e_a^B f_a^B + e_b^B f_b^B.$$

We then get

$$\begin{aligned} P_{ab}^{net} &= [\alpha_{ab} e_a^h + (1-\alpha_{ab}) e_b^h] f_{ab}^h \\ &+ [(1-\alpha_{ab}) e_a^u + \alpha_{ab} e_b^u] f_{ab}^u + [\beta_1 e_a^v + \beta_2 e_b^v] f_{ab}^v, \end{aligned} \tag{6.42}$$

where $\alpha_{ab} := \int_{Z_{ab}} \omega_a^h(z)\omega_{ab}^h(z)$, $\alpha_{ba} := \int_{Z_{ab}} \omega_b^h(z)\omega_{ab}^h(z)$, $\beta_1 := \int_{Z_{ab}} \omega_a^v(z)\omega_{ab}^v(z)$, $\beta_2 = \int_{Z_{ab}} \omega_b^v(z)\omega_{ab}^v(z)$. We use the above expression for identifying the port variables in the discretized interconnection structure. The flow variable corresponding to the mass density is f_{ab}^h and the effort variable is $\alpha_{ab} e_a^h + (1 - \alpha_{ab}) e_b^h$. Thus we define

$$\begin{aligned} e_{ab}^h &:= [\alpha_{ab} e_a^h + (1-\alpha_{ab}) e_b^h] \\ e_{ab}^u &:= [(1-\alpha_{ab}) e_a^u + \alpha_{ab} e_b^u] \\ e_{ab}^u &:= [\beta_1 e_a^v + \beta_2 e_b^v]. \end{aligned} \tag{6.43}$$

In addition to the properties of the zero and one forms in Proposition 6.1, we also have the following properties which are crucial in deriving the expression for power balance in the finite-dimensional case

Proposition 6.2. *Under the assumption that* $\omega_{ab}^v = \omega_{ab}^u$, *the constants* $\alpha_{ab}, \alpha_{ba}, \beta_1, \beta_2, c_1, c_2, c_3, c_4, c_1', c_2', c_3', c_4'$ *satisfy*

$$\begin{aligned} &\alpha_{ba} c_1 = \beta_1 c_1', \quad \alpha_{ba} c_3 = \beta_1 c_3', \quad \alpha_{ba} c_2 = \beta_2 c_1', \quad \alpha_{ba} c_4 = \beta_2 c_3', \\ &\alpha_{ab} c_1 = \beta_1 c_2', \quad \alpha_{ab} c_3 = \beta_1 c_4', \quad \alpha_{ab} c_2 = \beta_2 c_2', \quad \alpha_{ab} c_4 = \beta_2 c_4'. \end{aligned} \tag{6.44}$$

Proof. We know from Proposition 6.1 that

$$\omega_a^u(z) + \omega_b^u(z) = 1.$$

Hence, from satisfying of the compatibility conditions of (6.37,6.38,6.39) we have the following

$$(c_1 + c_2)\omega^u_{ab} = \tfrac{d(\omega_a+\omega_b)}{*\omega^h_{ab}}(\omega^v_a + \omega^v_b)$$

$$(c_3 + c_4)\omega^u_{ab} = \tfrac{d(\omega_a+\omega_b)}{*\omega^h_{ab}}(\omega^v_a + \omega^v_b)$$

$$(c_1' + c_2')\omega^v_{ab} = \tfrac{d(\omega_a+\omega_b)}{*\omega^h_{ab}};\ (c_3' + c_4')\omega^v_{ab} = \tfrac{d(\omega_a+\omega_b)}{*\omega^h_{ab}}.$$

using the above equalities the relations (6.44) can be easily proved. $\square$

Then, the net expression for power becomes

$$P^{net}_{ab} := f^h_{ab}e^h_{ab} + f^u_{ab}e^u_{ab} + f^v_{ab}e^v_{ab} - e^B_a f^B_a + e^B_b f^B_b. \tag{6.45}$$

Remark 6.3. Observe that the expression for energy balance is same as in the (h,u) case, see Equation (6.15). In this case we see that the additional port variables arising due to the velocity component v does not play any role. This property was also observed in the infinite-dimensional case in Chapter 2.

Now by substituting

$$e^h_a = e^B_a, e^h_b = e^B_b, e^u_a = f^B_a, e^u_b = f^B_b, e^v_a = f^B_v, e^v_b = e^v_b,$$

yields

$$\begin{bmatrix} -1 & 0 & 0 & \alpha_{ab} & \alpha_{ba} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \frac{c_2 v^a + c_4 v^b}{h_{ab}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^h_{ab} \\ e^u_{ab} \\ e^v_{ab} \\ e^B_a \\ e^B_b \\ e^v_b \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{ba} & \alpha_{ab} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_1 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{c_1 v^a + c_3 v^b}{h_{ab}} \\ 0 & 0 & 1 & -\frac{c_1' v^a + c_3' v^b}{h_{ab}} & -\frac{c_2' v^a + c_4' v^b}{h_{ab}} & 0 \end{bmatrix} \begin{bmatrix} f^h_{ab} \\ f^u_{ab} \\ f^v_{ab} \\ f^B_a \\ f^B_b \\ f^v_b \end{bmatrix}. \tag{6.46}$$

The above equation represents the spatially discretized interconnection structure, abbreviated as

$$D_{ab} = \{(f^{ab}, e^{ab}) \in \mathbb{R}^{12} : E_{ab}e_{ab} + F_{ab}f_{ab} = 0\}.$$

It can easily be shown that the above subspace D_{ab} is a Dirac structure with respect to the bilinear form

$$<< (f^{ab}_1, e^{ab}_1), (f^{ab}_2, e^{ab}_2) >>:=< e^{ab}_1, f^{ab}_2 > + < e^{ab}_2, f^{ab}_1 > . \tag{6.47}$$

6.2.2 Approximation of the energy part

For the discretization of the energy part we proceed as follows: The flow variables f_h, f_u and f_v and the energy variables h, u and v are one-forms. Since f_h, f_u and f_v are approximated by (6.33) and are related to h, u and v by (6.32), it is consistent to approximate h,u and v on Z_{ab} in the same way by

$$\begin{aligned} h(t,z) &= h_{ab}(t)\omega^h_{ab}(z) \\ u(t,z) &= u_{ab}(t)\omega^u_{ab}(z) \\ v(t,z) &= v_{ab}(t)\omega^u_{ab}(z), \end{aligned} \tag{6.48}$$

where

$$-\frac{dh_{ab}(t)}{dt} = f^h_{ab}(t), -\frac{du_{ab}(t)}{dt} = f^u_{ab}(t), -\frac{dv_{ab}(t)}{dt} = f^v_{ab}(t). \tag{6.49}$$

Here h_{ab} represents the total amount of water in the considered part of the canal and u_{ab}, v_{ab} the average velocities of the same part of the canal. The kinetic energy as a function of the energy variables u and v is given by

$$\int_{Z_{ab}} \frac{1}{2}[(*u(t,z))h(t,z)(*u(t,z)) + (*v(t,z))h(t,z)(*v(t,z))].$$

Approximation of the infinite-dimensional energy variables u and v by (6.49) means that we restrict the infinite-dimensional space of one-forms $\Omega^1(Z_{ab})$ to its one-dimensional subspace spanned by $\omega^h_{ab}, \omega^u_{ab}, \omega^v_{ab}$. This leads to the approximation of the kinetic energy of the considered part of the canal by

$$H^{u,v}_{ab}(h_{ab}, u_{ab}, v_{ab}) = \frac{1}{2}(C_1 h_{ab} u^2_{ab} + C_2 h_{ab} v^2_{ab}),$$

where

$$\begin{aligned} C_1 &= \int_{Z_{ab}} (*\omega^u_{ab}(z))\omega^h_{ab}(z) * \omega^u_{ab}(z) \\ C_2 &= \int_{Z_{ab}} (*\omega^v_{ab}(z))\omega^h_{ab}(z) * \omega^v_{ab}(z). \end{aligned}$$

Note that this is nothing else than the restriction of the kinetic energy function to the one dimensional subspace of $\Omega^1(Z_{ab})$. Similarly the potential energy is approximated by

$$H^h_{ab}(h_{ab}) = \frac{C_3}{2} g h^2_{ab},$$

where

$$C_3 = \int_{Z_{ab}} (*\omega^h_{ab}(z))\omega^h_{ab}(z).$$

Therefore, the total energy in the considered part of the canal is approximated by

$$
\begin{aligned}
H_{ab}(h_{ab}, u_{ab}, v_{ab}) &= H_{ab}^{u,v}(h_{ab}, u_{ab}, v_{ab}) + H_{ab}^{h}(h_{ab}) \\
&= \frac{1}{2}\left(C_1 h_{ab} u_{ab}^2 + C_2 h_{ab} v_{ab}^2 + C_3 g h_{ab}^2\right).
\end{aligned}
$$

Next, in order to describe the discretized dynamics, we equate the discretized effort variables $e_{ab}^h, e_{ab}^u, e_{ab}^v$ of the discretized interconnection structure defined in (6.43) with co-energy variables corresponding to the total approximated energy H_{ab} of the considered part of the canal

$$
\begin{aligned}
e_{ab}^h &= \tfrac{\partial H(h_{ab}, u_{ab}, v_{ab})}{\partial h_{ab}}(t) = \tfrac{1}{2}(C_1 u_{ab}^2 + C_2 v_{ab}^2) + C_3 g h_{ab} \\
e_{ab}^u &= \tfrac{\partial H(h_{ab}, u_{ab}, v_{ab})}{\partial u_{ab}}(t) = C_1 h_{ab} u_{ab} \\
e_{ab}^v &= \tfrac{\partial H(h_{ab}, u_{ab}, v_{ab})}{\partial v_{ab}}(t) = C_2 h_{ab} v_{ab}.
\end{aligned}
\tag{6.50}
$$

The equations (6.46) (the interconnected structure) together with (6.49),(6.50) represent a finite-dimensional model of the shallow water equations with a non-constant Stokes-Dirac structure. To sum up we have the following set of equations for a single lump of the finite-dimensional model

$$
\begin{aligned}
&-\frac{dh_{ab}}{dt} = hu\mid_a - hu\mid_b \\
&-\frac{du_{ab}}{dt} = \frac{1}{2}(u^2+v^2) + gh\mid_a - \frac{1}{2}(u^2+v^2) + gh\mid_a \\
&-\left(\frac{c_2 v_a + c_4 v_b}{h_{ab}} hv\mid_a + \frac{c_1 v_a + c_3 v_b}{h_{ab}} hv\mid_b\right) \\
&-\frac{dv_{ab}}{dt} = \frac{c_2' v_a + c_4' v_b}{h_{ab}} hu\mid_a + \frac{c_1' v_a + c_3' v_b}{h_{ab}} hu\mid_b \\
&\frac{1}{2}(C_1 u_{ab}^2 + C_2 v_{ab}^2) + C_3 g h_{ab} = \alpha_{ab}\left(\frac{1}{2}(u^2+v^2) + gh\mid_a\right) \\
&\qquad\qquad + \alpha_{ba}\left(\frac{1}{2}(u^2+v^2) + gh\mid_b\right) \\
&C_1 h_{ab} u_{ab} = \alpha_{ba}(hu\mid_a) + \alpha_{ab}(hu\mid_b) \\
&C_2 h_{ab} v_{ab} = \beta_1(hv\mid_a) + \beta_2(hv\mid_b).
\end{aligned}
\tag{6.51}
$$

Spatial discretization of the entire system

The canal is split into n parts. The ith part (S_{i-1}, S_i) is discretized as explained in the previous subsections, where $a = S_{i-1}$ and $b = S_i$. The resulting model consists of n submodels each of them representing a port-Hamiltonian system. Since a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system, the total discretized system is also a port-Hamiltonian system, whose interconnection structure is given by the composition of the n Dirac structures on (S_{i-1}, S_i), while the total Hamiltonian is the sum of individual Hamiltonians as

$$H(h,u) = \sum_{i=1}^{n} [C_{1i} h_l u_{S_{i-1},S_i} + C_{2i} h_l v_{S_{i-1},S_i} + C_3 g h^2_{S_{i-1},S_i}].$$

Here $h = (h_{S_0,S_1}, h_{S_1,S_2}, ..., h_{S_{n-1},S_n})^T$ are the discretized heights and $u = (u_{S_0,S_1}, u_{S_1,S_2}, ..., u_{S_{n-1}}, u_{s_n})$ and $v = (v_{S_0,S_1}, v_{S_1,S_2}, ..., v_{S_{n-1}}, v_{s_n})$ are the discretized velocities. The total discretized model still has two ports. The port $(f^B_{S_0}, e^B_{S_0}) = (f^B_0, e^B_0)$ is the incoming port and the port $(f^B_{S_n}, e^B_{Sn}) = (f^B_S, e^B_S)$ is the outgoing port, resulting in the energy balance of the discretized model

$$\frac{dH(h(t), u(t), v(t))}{dt} - e^B_0 f^B_0 + e^B_S f^B_S = 0.$$

Equation (6.40) for the ith part becomes $f^h_{S_{i-1},S_i}(t) = e^u_{S_{i-1}}(t) - e^u_{S_i}(t)$. Taking into account (6.49) and $e^u_{S_0} = f^B_0$, $e_{S_n} = f^B_S$, we have $\frac{dh(t)}{dt} = f^B_0 - f^B_S$, where $h := \sum\limits_{i=1}^{n} h_{S_{i-1},S_i}$ is the total mass (amount of water) in the canal, this represents mass conservation.

The input-state-output model

In this section we write the discretized system in the input state output model, which could help us further analyze the properties of the finite-dimensional model and compare it with the infinite-dimensional model. To simplify the model we use the following choices for the approximating zero and one forms. The zero-forms are approximated as constant density functions, i.e.

$$\omega^{h,u,v}_{ab} = \frac{1}{b-a},$$

and the zero-forms as linear splines, i.e.

$$\omega^{h,u,v}_{a} = \frac{b-z}{b-a}, \quad \omega^{h,u,v}_{b} = \frac{z-a}{b-a}.$$

This would result in the following values for the constants in (6.44).

$$\begin{aligned} \alpha_{ab} = \alpha_{ba} = \beta_1 = \beta_2 &= \frac{1}{2} \\ c_1 = c_1' = c_2 = c_2' &= -\frac{1}{2} \\ c_3 = c_3' = c_4 = c_4' &= \frac{1}{2}. \end{aligned}$$

We then have the following

$$\begin{bmatrix} f^h_{ab} \\ f^u_{ab} \\ f^v_{ab} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 2K \\ 0 & -2K & 0 \end{bmatrix} \begin{bmatrix} e^h_{ab} \\ e^u_{ab} \\ e^v_{ab} \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} f^B_a \\ -e^B_b \end{bmatrix}$$
$$\begin{bmatrix} e^B_a \\ f^B_b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} e^h_{ab} \\ e^u_{ab} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f^B_a \\ -e^B_b \end{bmatrix},$$

where

$$K = \frac{(v_a - v_b)}{h_{ab}}.$$

If we now apply the theory of Casimirs for an *autonomous* port-Hamiltonian system from Chapter 4 for a single lump, we see that the Casimirs are all functions $C(h, u, v)$ which satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 2K \\ 0 & -2K & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial C}{\partial h_{ab}} \\ \frac{\partial C}{\partial u_{ab}} \\ \frac{\partial C}{\partial v_{ab}} \end{bmatrix},$$

from the above equation we have

$$\frac{\partial C}{\partial u_{ab}} = 0$$
$$\frac{\partial C}{\partial h_{ab}} - \frac{(v_a - v_b)}{h_{ab}} \frac{\partial C}{\partial v} = 0. \tag{6.52}$$

This means that the Casimirs are independent of the u component of velocity which is consistent with the continuous case. Equation (6.52) could be seen as an analogue of (4.48), the solution of which would result in a class of functions which would be conserved quantities for the finite-dimensional model.

Remark 6.4 (Higher order approximation). : In the above discretization we approximated $*v$ in Equation (6.36) in the expression of the Stokes-Dirac structure (6.32), by a zero-form instead of using the approximation of v as a one-form. This is because throughout we considered the approximating one forms to be constant density functions (see Figure 6.1) and with such a choice the value of $\mathrm{d}(*\omega^v_{ab})$ in the discretized model would always be zero and hence the discretization procedure would fail. We can however use another approximations for the one-forms, instead of the constant density functions. These one-forms should be chosen such that not only $\mathrm{d}(*\omega^v_{ab}) \neq 0$, but they should also satisfy conditions (6.2) and the subsequent compatibility conditions. One such choice could be to choose the approximating one-forms as

$$\omega^{h,u,v}_{ab} = \frac{2z}{b^2 - a^2},$$

and the following choice for the zero-forms .

$$\omega_a^{h,u,v} = \frac{b^2 - z^2}{b^2 - a^2}, \quad \omega_b^{h,u,v} = \frac{z^2 - a^2}{b^2 - a^2}.$$

It can easily be seen that with the above choice $\mathrm{d}(*\omega_{ab}^v) \neq 0$ and they also satisfy the compatibility conditions. The implications of using such higher order approximations on the resulting finite-dimensional model remains to be seen.

6.3 Preliminary numerical results

The spatial discretization of one dimensional linear and nonlinear shallow water equations using port-Hamiltonian frame work are presented. The discretization typically consists of a set of ordinary differential and algebraic equations in which we seek for the solution of flows and efforts ("energy fluxes") numerically. We have attempted to solve the resulting ordinary differential equations using explicit time-stepping schemes like Euler-forward and Runge-Kutta methods. We found that these time-stepping schemes are numerically unstable and hence we use the Crank-Nicholson time-stepping scheme. Investigation on explicit time-stepping schemes and their stability analysis is beyond the scope of this study.

6.3.1 Harmonic wave type solution

We consider the following harmonic wave type solution of one dimensional linear shallow water equation in a domain $[0, L]$:

$$\begin{aligned} h(z,t) &= h_l + \eta(z,t), \quad \eta(z,t) = A\sin(kz + \omega t), \qquad \text{and} \\ u(z,t) &= \Big(\frac{-Agk}{\omega}\Big)\sin(kz + \omega t). \end{aligned} \tag{6.53}$$

where A is the amplitude, $k = 2\pi m/L$ the wave number, ω the actual frequency, H the mean free surface depth, $a^2 = gh_l$, $\omega^2 = a^2k^2$ the dispersion relation and m an integer. We have initialized the exact solution 6.53 in the linear numerical code with parameters $L = 1$, $m = 1$, $g = 1$, and $h_l = 1$; and simulated the waves for one time period $T = 2\pi/\omega$. Figure 6.3(a) shows the space-time profile of the free surface perturbation around the mean water depth from $t = 0$ to 1.0. The numerical discretization not only conserves mass but also energy as shown in Figure 6.3(b). Below we give an analytical proof of energy conservation in time for the discretized linear shallow water equations.

Proposition 6.5. *Consider the discretized linear shallow water equations (6.26), which for the $i-th$ lump would be as follows*

$$
\begin{aligned}
&-\frac{dh_{S_i,S_{i+1}}}{dt} = h_l u \mid_{S_i} - h_l u \mid_{S_{i+1}} \\
&-\frac{du_{S_i,S_{i+1}}}{dt} = gh \mid_{S_i} - gh \mid_{S_{i+1}} \\
&\frac{gh_{S_i,S_{i+1}}}{\Delta z} = \alpha_{S_i,S_{i+1}}(gh \mid_{S_i}) + \alpha_{S_{i+1},S_i}(gh \mid_{S_{i+1}}) \\
&\frac{h_l u_{S_i,S_{i+1}}}{\Delta z} = \alpha_{S_{i+1},S_i}(h_l u \mid_{S_i}) + \alpha_{S_i,S_{i+1}}(h_l u \mid_{S_{i+1}}).
\end{aligned}
\tag{6.54}
$$

In case of a uniform mesh Δz would be constant. The total energy is given by

$$H(h,u) = \frac{1}{\cancel{2}\Delta z}\sum_{i=1}^{n}[h_l u^2_{S_{i-1},S_i} + gh^2_{S_{i-1},S_i}]. \tag{6.55}$$

For a periodic boundary the efforts can be calculated by the following expression

$$
\begin{aligned}
e_{S_i} = {}& ... + f_{S_{i-3},S_{i-2}} - f_{S_{i-2},S_{i-1}} + f_{S_{i-1},S_i} + f_{S_i,S_{i+1}} - f_{S_{i+1},S_{i+2}} + ...; \\
& i = 1, ..., n.
\end{aligned}
\tag{6.56}
$$

This expression is obtained by solving the last two equations of (6.54) simultaneously. Therefore

$$e_{S_i} - e_{S_{i+1}} = ... - f_{S_{i-2},S_{i-1}} + f_{S_{i-1},S_i} - f_{S_{i+1},S_{i+2}} \cdots \tag{6.57}$$

where

$$e_{S_i} = \begin{bmatrix} gh \mid_{S_i} \\ h_l u \mid_{S_i} \end{bmatrix}; \;\; f_{S_i,S_{i+1}} = \begin{bmatrix} \frac{gh_{S_i,S_{i+1}}}{\Delta z} \\ \frac{h_l u_{S_i,S_{i+1}}}{\Delta z} \end{bmatrix}. \tag{6.58}$$

Time stepping scheme: As stated before, we use the Crank-Nicholson time stepping scheme. The time discretized equations take the form

$$
\begin{aligned}
h^{t_{n+1}}_{S_i,S_{i+1}} - h^{t_{n+1}}_{S_i,S_{i+1}} &= -\tfrac{\Delta t}{2}[(h_l u \mid^{t_{n+1}}_{S_i} - h_l u \mid^{t_{n+1}}_{S_{i+1}}) - (h_l u \mid^{t_n}_{S_i} - h_l u \mid^{t_n}_{S_{i+1}})] \\
u^{t_{n+1}}_{S_i,S_{i+1}} - u^{t_{n+1}}_{S_i,S_{i+1}} &= -\tfrac{\Delta t}{2}[(gh \mid^{t_{n+1}}_{S_i} - gh \mid^{t_{n+1}}_{S_{i+1}}) - (gh \mid^{t_n}_{S_i} - gh \mid^{t_n}_{S_{i+1}})].
\end{aligned}
\tag{6.59}
$$

In this case the energy of the system is conserved in time.

Proof. Substituting the expressions for efforts from Equation (6.57) we get

$$\begin{aligned}
&h^{t_{n+1}}_{S_i,S_{i+1}} - h^{t_{n+1}}_{S_i,S_{i+1}}(t_n) \\
&= -\frac{h_l \Delta t}{2}[(... - u^{t_{n+1}}_{s_{i-2},S_{i-1}} + u^{t_{n+1}}_{S_{i-1},S_i} - u^{t_{n+1}}_{S_{i+1},S_{i+2}} + ...) \\
&- (... - u^{t_n}_{s_{i-2},S_{i-1}} + u^{t_n}_{S_{i-1},S_i} - u^{t_n}_{S_{i+1},S_{i+2}} + ...)],
\end{aligned} \tag{6.60}$$

$$\begin{aligned}
&u^{t_{n+1}}_{S_i,S_{i+1}} - u^{t_{n+1}}_{S_i,S_{i+1}} \\
&= -\frac{g \Delta t}{2}[(... - h^{t_{n+1}}_{s_{i-2},S_{i-1}} + h^{t_{n+1}}_{S_{i-1},S_i} - h^{t_{n+1}}_{S_{i+1},S_{i+2}} + ...) \\
&- (... - h^{t_n}_{s_{i-2},S_{i-1}} + h^{t_n}_{S_{i-1},S_{i+2}} - h^{t_n}_{S_{i+1},S_{i+2}} + ...)].
\end{aligned} \tag{6.61}$$

Multiplying the first part of the above equation by $g\frac{h_{S_i,S_{i+1}}(t_{n+1}) + h_{S_i,S_{i+1}}(t_n)}{2\Delta z}$, we can write the the total discretized potential energy as

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{g\left[(h^{t_{n+1}}_{S_i,S_{i+1}})^2 - (h^{t_n}_{S_i,S_{i+1}})^2\right]}{2\Delta z} \\
&= -\sum_{i=1}^{n}\left\{\frac{g h_l \Delta t}{(\Delta z)^2}\ \{... - (u^{t_{n+1}}_{S_{i-2},S_{i-1}} + u^{t_n}_{S_{i-2},S_{i-1}})(h^{t_{n+1}}_{S_i,S_{i+1}} + h^{t_n}_{S_i,S_{i+1}})\right. \\
&+ (u^{t_{n+1}}_{S_{i-1},S_i} + u^{t_n}_{S_{i-1},S_i})(h^{t_{n+1}}_{S_i,S_{i+1}} + h^{t_n}_{S_i,S_{i+1}}) \\
&\left. -(u^{t_{n+1}}_{S_{i+1},S_{i+2}} + u^{t_n}_{S_{i+1},S_{i+2}})(h^{t_{n+1}}_{S_i,S_{i+1}} + h^{t_n}_{S_i,S_{i+1}}) + ...\right\}.
\end{aligned}$$

Similarly, by multiplying the second part of Equation(6.61) by $h_l\frac{u_{S_i,S_{i+1}}(t_{n+1}) + u_{S_i,S_{i+1}}(t_n)}{2\Delta z}$ the expression for the kinetic energy takes the following form

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{h_l\left[(u^{t_{n+1}}_{S_i,S_{i+1}})^2 - (u^{t_{n+1}}_{S_i,S_{i+1}})^2\right]}{(\Delta z)^2} \\
&= -\sum_{i=1}^{n}\left\{\frac{g h_l \Delta t}{(\Delta z)^2}\ \{... - (h^{t_{n+1}}_{s_{i-2},S_{i-1}} + h^{t_n}_{s_{i-2},S_{i-1}})(u^{t_{n+1}}_{S_i,S_{i+1}} + u^{t_n}_{S_i,S_{i+1}})\right. \\
&+ (h^{t_{n+1}}_{S_{i-1},S_i} + h^{t_n}_{S_{i-1},S_i})(u^{t_{n+1}}_{S_i,S_{i+1}} + u^{t_n}_{S_i,S_{i+1}}) \\
&\left. -(h^{t_{n+1}}_{S_{i+1},S_{i+2}} + h^{t_n}_{S_{i+1},S_{i+2}})(u^{t_{n+1}}_{S_i,S_{i+1}} + u^{t_n}_{S_i,S_{i+1}}) + ...\right\}.
\end{aligned} \tag{6.62}$$

Since we have a periodic domain and also since the summation is over the entire domain, we can rewrite the right hand side of (6.62), by interchanging the indices, as follows

$$
\begin{aligned}
&\sum_{i=1}^{n} \frac{g\left[(h^{t_{n+1}}_{S_i,S_{i+1}})^2-(h^{t_n}_{S_i,S_{i+1}})^2\right]}{2\Delta z}\\
&=-\sum_{i=1}^{n}\left\{\frac{gh_l\Delta t}{(\Delta z)^2}\ \{...-(u^{t_{n+1}}_{S_i,S_{i+1}}+u^{t_n}_{S_i,S_{i+1}})(h^{t_{n+1}}_{S_{i+2},S_{i+3}}+h^{t_n}_{S_{i+2},S_{i+3}})\right.\\
&+(u^{t_{n+1}}_{S_i,S_{i+1}}+u^{t_n}_{S_i,S_{i+1}})(h^{t_{n+1}}_{S_{i+1},S_{i+2}}+h^{t_n}_{S_{i+1},S_{i+2}})\\
&\left.-(u^{t_{n+1}}_{S_i,S_{i+1}}+u^{t_n}_{S_i,S_{i+1}})(h^{t_{n+1}}_{S_{i-1},S_i}+h^{t_n}_{S_{i-1},S_i})+...\right\}. \qquad (6.63)
\end{aligned}
$$

Now adding (6.62) and (6.63) we have that, the rate of change of total energy

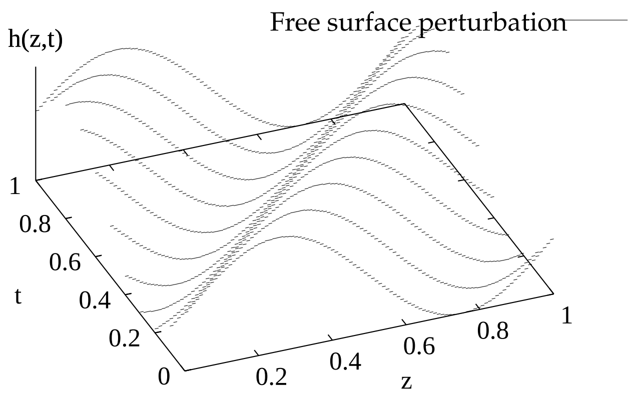
$$
\sum_{i=1}^{n}\frac{h_l\left[(u^{t_{n+1}}_{S_i,S_{i+1}})^2-(u^{t_{n+1}}_{S_i,S_{i+1}})^2\right]}{(\Delta z)^2}+\frac{g\left[(h^{t_{n+1}}_{S_i,S_{i+1}})^2-(h^{t_n}_{S_i,S_{i+1}})^2\right]}{2\Delta z}=0, \quad (6.64)
$$

which implies that

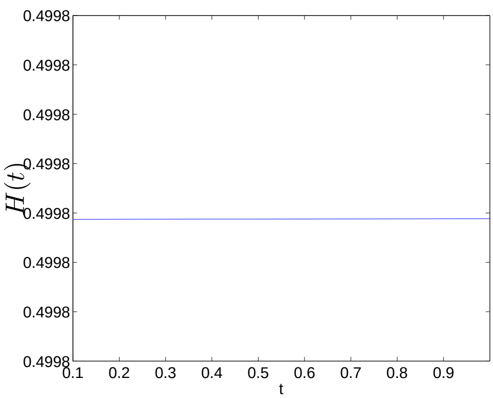
$$
H^{t_{n+1}}_{S_i,S_{i+1}}-H^{t_n}_{S_i,S_{i+1}}=0,
$$

and hence we prove that the total energy of the discretized system is conserved in time. □

We have also developed the numerical scheme for the nonlinear shallow water equations with Crank-Nicholson time discretization. The resulting nonlinear algebraic equations are solved iteratively using Newton-Raphson method. To test the numerical scheme, we first initialize the linear harmonic wave solution at $t=0$ under a low and high amplitudes. Figure 6.4(a) shows the space-time profile of the water depth h from $t=0$ to 1.0. Because of nonlinearity, the high amplitude waves start steepening as shown in Figure 6.4(b).

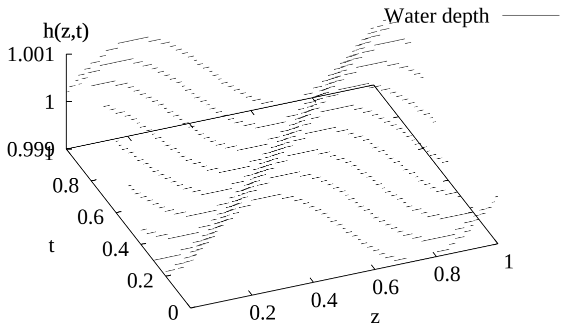


(a)

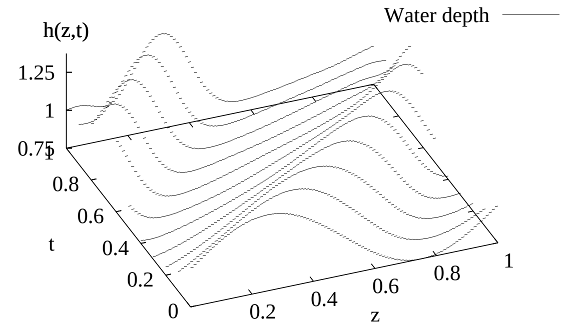


(b)

Figure 6.3: a) Space-time profile of the free surface perturbation obtained from the numerical scheme of the port-Hamiltonian discretization of the linear shallow water equations with amplitude $A = 0.01$, $\Delta z = 1/101$ and $\Delta t = T/10$. b) Plot of the energy $H(t) = \int_Z \frac{1}{2}(h_l u^2 + gh^2 \mathrm{d}z$ versus time.



(a)



(b)

Figure 6.4: Space-time profile of the water depth obtained from the numerical scheme of the port-Hamiltonian discretization of the nonlinear shallow water equations with $\Delta z = 1/101$, $\Delta t = T/10$, amplitude (a) $A = 0.001$ and (b) $A = 0.25$.

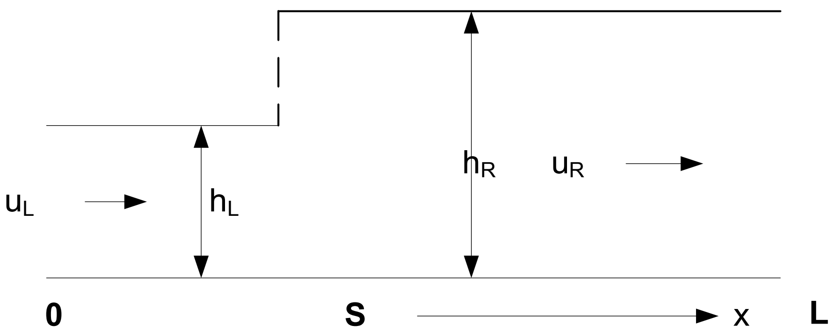


Figure 6.5: Shock wave

6.4 Discussion: Modeling procedure enabling to capture shocks

Consider the shallow-water equations (2.64). Mathematically, this is a system of conservation laws. It represents conservation of mass and conservation of particle speed u. The velocity conservation law is, however physically meaningless. System (2.64) is mathematically conservative but physically non-conservative. We shall see next, that the we can use (2.60) to model the shallow water equations, provided the solutions are smooth. Assume that the solution to (2.64) consists of a shock wave of speed S_i. The following Rankine-Hugoniot conditions then apply for discontinuous solutions of hyperbolic conservation laws [56]

$$F(U_R) - F(U_L) = S_i(U_R - U_L).$$

In case of the shallow water equations (2.64)

$$U = [h \;\; u]^T$$
$$F(U) = [\frac{1}{2}u^2 + gh \quad hu]^T.$$

Consider a shock wave (Figure 6.5) in which the state ahead of the shock is given by the variables (h_R, u_R) and the state before the shock is given by (h_L, u_L). The speed of the shock wave is given by

$$S_1 = u_R + \sqrt{\frac{2gh_L^2}{h_L + h_R}}.$$

The shallow water equations in this form are often referred to as equations in a *quasi-linear* form.

More generally, the shallow water equations are written in a *conservative* form as

$$\partial_t h + \partial_x(hu) = 0$$
$$\partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = 0,$$

which express the physical conservation laws of mass and momentum. In this case the shock wave has a shock speed (for a detailed analysis of shock waves we refer to [56])

$$S_2 = u_R + \sqrt{\frac{1}{2}g\frac{(h_L + h_R)h_L}{h_R}}$$

By comparing the two shock speeds S_1 and S_2 it is found that

$$S_1 \leq S_2$$

This results from the inequality

$$0 \leq (h_L - h_R)^2$$

Equality of the shock speeds holds only if the shock is trivial, when $h_L = h_R$, that is when the solution is smooth. This clearly says that the formulation of the shallow water equations as in Equation (2.64) does not admit shocks in a physical sense. It would be of interest to have a modeling procedure, in the port-Hamiltonian framework, which admits shocks or discontinuities in the solution.

One way to look at it could be to view the whole system as composition of two infinite-dimensional port-Hamiltonian systems with interconnection constraints in such a way that they take into consideration the physical implications of the shock, namely

$$h_L u_L = h_R u_R$$
$$h_L u_L^2 + \frac{1}{2}gh_L^2 = h_R u_R^2 + \frac{1}{2}gh_R^2.$$

However, to obtain a general modeling procedure still remains an open issue.

7

Conclusions and future research

"How many times can you subtract 7 from 83, and what is left afterwards? You can subtract it as many times as you want, and it leaves 76 every time." - Anonymus.

7.1 Conclusions

In this thesis we have analyzed properties of interconnected physical systems, from various domains, in the framework of port-Hamiltonian systems. It is shown how this framework can serve as a powerful tool for modeling and control of complex systems. An important property of port-Hamiltonian systems is that a power-conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system and hence this framework has proven to be well suited for modeling and control of energy-conserving physical systems using the network approach.

In this chapter we highlight the important results obtained in the thesis and we give some directions for future research.

7.2 Contributions of the thesis

The contributions of the thesis can be summarized as follows:

- The port-Hamiltonian formulation has been used to model flow of water though a canal, modeled by the shallow water equations in a *quasi-linear* form. We also presented an extension of the 1-D case which is interesting from the mathematical point of view. In this case the Stokes-Dirac structure is *non constant* as it depends on the energy variables.

- We define the notion of composition of a Dirac structure and a resistive relation. This composition helps us to study interconnections of port-Hamiltonian systems with dissipation, both from the finite and infinite-dimensional domains. We also extend this to the case of infinite-dimensional systems with dissipation.

- The set of achievable Dirac structures in the composition of a given plant Dirac structure with a to-be-designed controller Dirac structure has been extended to systems with dissipation. This has also been successfully applied to the case of infinite-dimensional systems with dissipation and to mixed finite and infinite-dimensional systems.

- The set of achievable Dirac structures leads to a characterization of the set of achievable Casimirs of the closed-loop system. The characterization of the set of achievable Casimirs helps us to study the implications of Casimirs on control of port-Hamiltonian systems. In the case of finite-dimensional systems with dissipation we prove, under certain conditions, that if a function is a Casimir for a given resistive relation it is a Casimir for all resistive relations.

- The theory of achievable Casimirs was used to study the problem of stabilization of a given system by generating Casimirs in the extended state space. It has been shown, with an example of an electromechanical system, how with the help of new passive outputs we can overcome the dissipation obstacle and design stabilizing controllers for the system. The Energy-Casimir method has also been applied towards obtaining some preliminary results on control of fluid dynamical systems..

- We have applied finite element methods in the port-Hamiltonian framework for spatial discretization of the shallow water equations. The resulting system is a finite-dimensional port-Hamiltonian system which retains all the properties of its infinite-dimensional analogue. A preliminary extension to the case of a non-constant Stokes-Dirac structure is also presented.

7.3 Recommendations for future work

7.3.1 Modeling of shallow water equations

As discussed in Section 6.4, the shallow water equations modeled as port-Hamiltonian systems in Chapter 2 do not admit shocks or discontinuities. A general modeling procedure, in the port-Hamiltonian framework, which takes into account discontinuities remains a matter of investigation.

7.3.2 Control of canal systems

In Chapter 5, we have presented some preliminary results on control of fluid dynamical systems of stabilizing flow through a canal (or a series of canals) at a desired height and zero velocity. In this case, to shape the energy we make use of one of the conservation laws of the system, that is the mass conservation. However, if we can use other conserved quantities, say for example the momentum hu, then we can stabilize the system not only at a desired height but also at a desired velocity by generating Lyapunov functions of the form

$$V = \frac{1}{2}[h(u-\bar{u})^2 + g(h-\bar{h})^2]$$

Such type of Lyapunov functions have also been derived in [8] and to arrive at such functions from the Hamiltonian point of view remains an open issue.

7.3.3 Interconnections in the mixed case

In Section 3.3.1 we have presented an extension of interconnections of mixed finite and infinite-dimensional systems, by presenting an example for the 2-D case and then generalized it to an infinite-dimensional system with an n-dimensional spatial domain. The interconnection constraints (3.34) hold for classes of systems where one of the boundary variables is a zero-form (which is the same as a function that takes value at points). Examples of such a case are the 3-D fluid flow, the n-D wave equation etc. This is however not the case when none of the boundary variables is a function, as in the case of Maxwell's equations where the boundary variables are the electric field intensity and the magnetic field intensity both being one-forms. To interconnect systems of this sort through the boundary with finite-dimensional systems remains an open issue.

7.3.4 Electromechanical systems

A possible way to overcome the dissipation obstacle is to generate new passive outputs which enables swapping of the damping. An example of stabilizing an electromechanical system with constant inputs, resulting in forced a equilibrium, was presented in Section 5.1.2. This method is not applicable to a general class of electromechanical systems and the case of the capacitor microphone could be treated as a special case. Based on the power shaping techniques introduced in [37] and successfully applied to electrical circuits, one might naturally be tempted to apply such techniques also to electromechanical systems. An important point to observe in the case of electromechanical systems, unlike electrical circuits, is that in general the Hamiltonian cannot be split into electrical and mechanical parts. To explore new passivity properties, which would help in solving control problems, for a general class of electromechanical systems remains an open issue.

7.3.5 Spatial discretization of infinite-dimensional systems

We have presented in Chapter 6 some preliminary results towards the spatial discretization of the shallow water equations. One of the issues which remain open is to explicitly relate the conserved quantities in the finite-dimensional case to the infinite-dimensional case for the huv formulation. The extension to the higher dimensional spatial domain case is certainly a matter of investigation. In the numerical results, it has been seen that various explicit time-stepping schemes have proven to be numerically unstable. Investigation on various time-stepping schemes and their stability analysis remains an open issue.

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